Hodge filtration and birational geometry I

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CIRM, April 14, 2022

Overview

- Joint work with **Mircea Mustață**.

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- Theme: (Hodge) D-modules can be brought into play when studying some of the most basic questions about closed subvarieties of smooth complex varieties.

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- Main tool: Hodge filtration on localizations, or more generally on local cohomology

Local cohomology

Set-up: • $Z \subseteq X$ closed (reduced) subscheme, X smooth **complex** variety.

• Have **local cohomology sheaves** $\mathcal{H}^q_Z\mathcal{O}_X$ for $q \geq 0$; obtained by taking higher derived functors of $\mathcal{H}^0_Z\mathcal{O}_X = \underline{\Gamma_Z}\mathcal{O}_X$ = subsheaf of local sections with support in Z.

(Locally: if $I \subseteq R$, then $H^0_I R = \{r \in R \mid I^k \cdot r = 0 \text{ for some } k \geq 0\}$.)

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(Locally: if $I \subseteq R$, then $H^0_I R = \{r \in R \mid I^k \cdot r = 0 \text{ for some } k \geq 0\}$.)

• Equivalently, difference between $\mathcal{O}_X$ and $\mathbf{R} j_* \mathcal{O}_U$, with $j \colon U = X \smallsetminus Z \hookrightarrow X$:

- $0 \to 0 = \mathcal{H}^0_Z \mathcal{O}_X \to \mathcal{O}_X \to j_* \mathcal{O}_U \to \mathcal{H}^1_Z \mathcal{O}_X \to 0$
- $R^{q-1} j_* \mathcal{O}_U \simeq \mathcal{H}^q_Z \mathcal{O}_X$ for $q \geq 2$

Local cohomology

Example: $Z = D = \textit{hypersurface} \implies$ only $\mathcal{H}^1_D \mathcal{O}_X \neq 0$, and have a SES:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(*D) \to \mathcal{H}^1_D \mathcal{O}_X \to 0$$

where $\mathcal{O}_X(*D) = j_* \mathcal{O}_U =$ rational functions with poles along D, with $j \colon U \hookrightarrow X, \quad U = X \smallsetminus D$.

- Locally, if $R = \mathcal{O}_X(V)$ and $D = (f = 0)$, this is $H^1_{(f)} R = R_f / R$.

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- $\mathcal{O}_X$ and $\mathcal{O}_X(*D)$ have natural $\mathcal{D}_X$-**module structure**: locally R and R_f, and differential operators act by the quotient rule; hence so does $\mathcal{H}^1_D \mathcal{O}_X$.

Local cohomology

- In general similar interpretation in terms of localization: say Z defined locally by $I = (f_1, \ldots, f_s) \subset R = \mathcal{O}_X(V)$. For $J \subset \{1, \ldots, s\}$ denote $f_J := \prod_{j \in J} f_j$.

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- $\exists$ Čech-type complex (of $\mathcal{D}_X$-modules)

$$C^\bullet : \quad 0 \to C^0 \to C^1 \to \cdots \to C^s \to 0$$

with

$$C^p := \bigoplus_{|J|=p} R_{f_J},$$

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- $r = 1$: $\quad 0 \to R \to R_f \to 0$
- $r = 2$: $\quad 0 \to R \to R_{f_1} \oplus R_{f_2} \to R_{f_1 f_2} \to 0$

Local cohomological dimension

Consequence: $\mathcal{H}^q_Z \mathcal{O}_X = 0$ for $q >$ *minimal* number of local defining equations for Z

- The **local cohomological dimension** of Z in X is

$$\mathrm{lcd}_X(Z) := \max\{q \mid \mathcal{H}^q_Z \mathcal{O}_X \neq 0\}.$$

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- Also have: $\min\{q \mid \mathcal{H}^q_Z \mathcal{O}_X \neq 0\} = \mathrm{codim}_X Z =: r.$

- Example: If Z is a local complete intersection (**LCI**), then only $\mathcal{H}^r_Z \mathcal{O}_X \neq 0$.

But can also have non-LCI subvarieties with $\mathrm{lcd}_X(Z) = r$. (E.g. most varieties with quotient singularities.) In general $\mathrm{lcd}_X(Z)$ rather mysterious; more later.

Hodge module structure

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Example: The case of a *hypersurface* $Z = D \subset X$ is relatively well understood:

- The $\mathcal{D}_X$-module $\mathcal{O}_X(*D)$ underlies the MHM $j_*\mathbf{Q}^H_U[n]$, i.e. the push-forward of the trivial Hodge module via $j\colon U = X \smallsetminus D \hookrightarrow X$; in particular comes with **Hodge filtration** $F_\bullet\mathcal{O}_X(*D)$.

- Puts MHM structure on local cohomology, since

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(*D) \to \mathcal{H}^1_D\mathcal{O}_X \to 0$$

Arbitrary codimension

- Analogously, for arbitrary $Z \subset X$ the complex

$$C^\bullet : \quad 0 \to C^0 \to C^1 \to \cdots \to C^s \to 0$$

is a complex of MHMs $\Longrightarrow$ get MHM structure on each $\mathcal{H}^q_Z \mathcal{O}_X$, in particular

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- Right $\mathcal{D}_X$-module version: $\quad \mathcal{H}^q_Z \omega_X \simeq \mathcal{H}^q_Z \mathcal{O}_X \otimes \omega_X \quad$ and $\quad F_k \mathcal{H}^q_Z \omega_X$.

Properties of the Hodge filtration

List of formal properties of the Hodge filtration on local cohomology, as consequences of **mixed Hodge module theory** and **birational geometry**:

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1) If $Z \subseteq X$ is **smooth** of codimension r, then

$$F_k\mathcal{H}^r_Z(\mathcal{O}_X) = O_k\mathcal{H}^r_Z(\mathcal{O}_X) := \{u \in \mathcal{H}^r_Z(\mathcal{O}_X) \mid I_Z^{k+1}u = 0\} \quad (\text{``order filtration''})$$

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- Concretely, if Z defined locally by $I = (x_1, \ldots, x_r) \subset R$, then

$$H^r_I(R) \simeq \mathrm{Coker}\Big(\bigoplus_{i=1}^{r} R_{x_1 \cdots \widehat{x_i} \cdots x_r} \to R_{x_1 \cdots x_r}\Big).$$

But we understand the Hodge filtration on localization along simple normal crossing divisors:

$$F_k R_{x_1 \cdots x_p} = F_k \mathcal{D}_X \cdot \frac{1}{x_1 \cdots x_p}$$

Properties: birational interpretation

2) **Birational interpretation.**

Let $f\colon Y \to X$ be a **log resolution** of (X, Z), assumed isom. over $U = X \smallsetminus Z$.
Denote $E = f^{-1}(Z)_{\mathrm{red}}$, an SNC divisor.

$$\begin{array}{ccc} V = Y \smallsetminus E & \longrightarrow & Y \\ \downarrow \simeq & & \downarrow f \\ U = X \smallsetminus Z & \xrightarrow{\ j\ } & X \end{array}$$

Say $q \ge 2$; have: $\quad (\mathcal{H}^q_Z \omega_X, F) \simeq R^{q-1} j_+(\omega_U, F) \simeq \mathcal{H}^{q-1} f_+(\omega_Y(*E), F).$

How do we compute RHS in practice?

Properties of the Hodge filtration

- $\exists$ complex of $f^{-1}\mathcal{D}_X$-modules on Y:

$$A^\bullet: \quad 0 \to f^*\mathcal{D}_X \to \Omega^1_Y(\log E) \otimes f^*\mathcal{D}_X \to \cdots \to \omega_Y(E) \otimes f^*\mathcal{D}_X \to 0$$

filtered by complexes of $\mathcal{O}_Y$-modules

$$F_k A^\bullet: \quad 0 \to f^*F_{k-n}\mathcal{D}_X \to \Omega^1_Y(\log E) \otimes f^*F_{k-n+1}\mathcal{D}_X \to \cdots \to \omega_Y(E) \otimes f^*F_k\mathcal{D}_X \to 0$$

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- Turns out that $\mathcal{H}^q_Z\omega_X \simeq R^{q-1}f_*A^\bullet$; filtration on $\mathcal{H}^q_Z\omega_X$ a priori given by

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- **Crucial Hodge-theoretic point** (Saito's strictness): φ_k is injective! (Generalized version of the E_1-degeneration of the Hodge-to-de Rham spectral sequence.)

Properties: birational interpretation

Conclusion: $$F_k \mathcal{H}^q_Z \omega_X \simeq R^{q-1} f_* F_k A^\bullet \qquad (q \geq 2,\ k \geq 0).$$

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Examples:

- $\mathbf{k = 0}$: $F_0 \mathcal{H}^q_Z \omega_X \simeq R^{q-1} f_* \omega_Y(E)$ ($\simeq R^{q-1} f_* \omega_E$ by Grauert-Riemenschneider).

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- $\mathbf{k = 1}$: $F_1 \mathcal{H}^q_Z \omega_X \simeq R^{q-1} f_* \left[\Omega^{n-1}_Y(\log E) \to \omega_Y(E) \otimes f^* F_1 \mathcal{D}_X \right].$

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- **Intuition**: "niceness" of (filtration on) $\mathcal{H}^q_Z \omega_X$ corresponds to lots of vanishing of higher direct images $R^p f_* \Omega^q_Y(\log E)$. (More on this later.)

Properties of the Hodge filtration

3) **Comparison with Ext filtration.** Combine with another well-known description of local cohomology:

$$\mathcal{H}^q_Z \omega_X \simeq \varinjlim_k \mathcal{E}xt^q\big(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, \omega_X\big).$$

Define $\quad E_k \mathcal{H}^q_Z \omega_X := \mathrm{Im}\, [\mathcal{E}xt^q\big(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, \omega_X\big) \to \mathcal{H}^q_Z \omega_X].$

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- Example: If $Z = D$ hypersurface, $q = 1$ and have (Saito)

$$F_k \mathcal{H}^1_D \omega_X \subseteq E_k \mathcal{H}^1_D \omega_X \simeq \omega_X((k+1)D)/\omega_X,$$

i.e. Hodge filtration is contained in "pole order filtration". Equivalently

$$F_k \omega_X(*D) = \omega_X\big((k+1)D\big) \otimes I_k(D),$$

with $I_k(D) =$ **k-th Hodge ideal** of D; rich theory.

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- General idea: think of E_k as the analogue of the pole order filtration (indeed equal to filtration O_k when Z is LCI); compare F_k and E_k.

Properties of the Hodge filtration

Facts: (i) Always have $F_0\mathcal{H}^q_Z\omega_X \subseteq E_0\mathcal{H}^q_Z\omega_X$, for all q; because of birational description, equivalent to injectivity of natural map

$$R^{q-1}f_*\omega_E \longrightarrow \mathcal{E}xt^q(\mathcal{O}_Z, \omega_X).$$

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• Well-known for a hypersurface D: $F_0 = E_0 \iff (X, D)$ log canonical $\iff D$ Du Bois. In general:

Theorem

If Z is Du Bois, then $F_0\mathcal{H}^q_Z\omega_X = E_0\mathcal{H}^q_Z\omega_X$ for all q. Converse also true if we assume Z Cohen-Macaulay (but not in general).

Properties of the Hodge filtration

(ii) If $Z \subseteq X$ is **LCI** of codimension r, then for all k:

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The **singularity index** of Z is

$$p(Z) := \max \{k \mid F_k = E_k\}.$$

(Convention: $p(Z) = -1$ if it never happens.)

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Theorem

If Z is singular, then $p(Z) \leq \frac{\mathrm{codim}_Z(Z_{\mathrm{sing}})-1}{2} \leq \frac{\dim Z - 1}{2}$.

Properties of the Hodge filtration

4) **Measure of singularities.** Based on the above, when Z is LCI:

- Z is smooth $\iff p(Z) = \infty$ (i.e. $F_k = E_k$ for all $k \geq 0$).
- $p(Z) \geq 0 \iff Z$ has Du Bois singularities. (More generally $p(Z) \geq p \iff Z$ has p-Du Bois singularities; see Mircea's talk.)
- $p(Z) \geq 1 \implies Z$ has rational singularities (Chen-Dirks-Mustață-Olano)
- **General conjecture:** $p(Z) = \max \{[\widetilde{\alpha}(Z)] - r, -1\}$.

  • Here $\widetilde{\alpha}(Z)$ = negative of the greatest root of the reduced Bernstein-Sato polynomial $\tilde{b}_Z(s) = b_Z(s)/(s+r)$ studied by Budur-Mustață-Saito.

  • Known when Z is a hypersurface, via connection with V-filtration due to Saito. (Main tool for understanding the *minimal exponent* of a hypersurface.)

5) **Vanishing theorem.** General Kodaira-Saito vanishing theorem for mixed Hodge module implies:

Corollary

If X is projective and L is an ample line bundle on X, then

$$\mathbf{H}^i\big(X, \mathrm{gr}_F^k \mathrm{DR}_X(\mathcal{H}_Z^q \mathcal{O}_X) \otimes L\big) = 0, \quad \forall\ i > 0, q \geq 0, k \in \mathbb{Z}.$$

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- Example: For $k = -n$, using birational interpretation, get

$$H^i(X, R^{q-1} f_* \omega_Y(E) \otimes L) = 0 \quad \forall i > 0.$$

Combination of vanishing theorems of Kollár, Fujino, Nadel.

Vanishing for Hodge ideals

- Example: Recall that when $Z = D$ is a hypersurface, for all $k \geq 0$ we have

$$F_k \omega_X(*D) = \omega_X\big((k+1)D\big) \otimes I_k(D).$$

If X projective and D ample, then under mild assumptions

$$H^i(X, \omega_X((k+1)D) \otimes I_k(D)) = 0 \quad \forall i > 0.$$

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- Combine with another consequence of the birational interpretation:

$$\mathrm{mult}_x(D) \geq \frac{n+p}{k+1} \implies I_k(D)_x \subseteq \mathfrak{m}_x^p.$$

Geometric application of Hodge ideals

"Generic" case of a folklore conjecture on theta divisors; problem with long history.

Theorem

Let (A, Θ) be a principally polarized abelian variety of dimension g. If Θ has isolated singularities, then

$$\mathrm{mult}_x(\Theta) \le \frac{g+1}{2} \quad \text{for all} \quad x \in \Theta.$$

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Moreover, equality can hold for at most one point.

Idea: use properties above for the Hodge ideal $I_1(\Theta)$.

- if $\mathrm{mult}_x(\Theta) \geq \frac{g+2}{2}$, then $I_1(\Theta) \subseteq \mathfrak{m}_x^2$.
- $H^1(A, \mathcal{O}_A(2\Theta) \otimes I_1(\Theta)) = 0$.

Geometric application of Hodge ideals

- $\mathfrak{m}_x^2/I_1(\Theta)$ is 0-dimensional, plus translation trick, gives

$$H^1(A, \mathcal{O}_A(2\Theta) \otimes \mathfrak{m}_y^2) = 0, \quad \forall y \in A.$$

- Geometrically this means that the linear system $|2\Theta|$ separates tangent vectors.

- **Contradiction**, since the map $A \to \mathbf{P}^N$ given by $|2\Theta|$ (the **Kummer map**) is ramified at the 2-torsion points.

Properties: generation level

6) **Generation level of the Hodge filtration.**

Definition: The filtration $F_\bullet\mathcal{M}$ on a $\mathcal{D}$-module $\mathcal{M}$ is **generated at level** k if

$$F_k\mathcal{M} \cdot F_\ell\mathcal{D}_X = F_{k+\ell}\mathcal{M} \quad \text{for all} \quad \ell \geq 0.$$

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- Birational characterization leads to:

Theorem

If $\mathcal{H}^p_Z\omega_X = 0$ for $p > q$, then the Hodge filtration on $\mathcal{H}^q_Z\omega_X$ is generated at level k if and only if

$$R^{q-1+i}f_*\,\Omega^{n-i}_Y(\log E) = 0 \quad \text{for all} \quad i > k.$$

(Hence always generated at level $n-q$.)

Local cohomological dimension

Application: **Characterization of local cohomological dimension**

- Recall that

$$\mathrm{lcd}_X(Z) := \max\{k \geq 0 \mid \mathcal{H}^k_Z\mathcal{O}_X \neq 0\}.$$

Studied by **Hartshorne, Ogus, Peskine-Szpiro**, then many others in CA. Over $\mathbb{C}$, complete **topological** characterization by Ogus in terms of algebraic de Rham/singular cohomology.

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Our strategy: To show $\mathcal{H}^q_Z \omega_X = 0$, verify two things:

- $F_0 \mathcal{H}^q_Z \omega_X = 0$.
- $F_\bullet \mathcal{H}^q_Z \omega_X$ is generated at level 0.

Local cohomological dimension

- Result on the generation level of $F_\bullet \mathcal{H}^q_Z \omega_X$ plus the description of $F_0 \mathcal{H}^q_Z \omega_X \implies$ alternative **algebraic/holomorphic** characterization in terms of a finite collection of coherent sheaves:

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Theorem

The following are equivalent:

1. $\mathrm{lcd}_X(Z) \le n - k$.
2. $R^{j+i} f_* \, \Omega_Y^{n-i}(\log E) = 0$ *for all* $j \ge n-k$ *and all* $i \ge 0$.

Examples: (Note: I ignore the $R^n f_*(\cdots)$, since they clearly vanish.)

1) We have:

$$\mathrm{lcd}_X(Z) \le n-1 \iff R^{n-1} f_* \omega_Y(E) = 0.$$

Not hard to check that the vanishing holds $\iff$ Z has no isolated points.

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Corollary (Hartshorne-Lichtenbaum Thm., smooth vars.)

$\mathrm{lcd}_X(Z) \le n-1 \iff$ *Z has no isolated points.*

Local cohomological dimension

2) We have

$$\mathrm{lcd}_X(Z) \le n-2$$

i.e $\mathcal{H}^{n-1}_Z\omega_X = \mathcal{H}^n_Z\omega_X = 0$, if and only if:

- $R^{n-1}f_*\Omega^{n-1}_Y(\log E) = 0$ (Non-obvious fact: this always holds.)
- $R^{n-1}f_*\omega_Y(E) = 0$ (Holds by 1).)
- $R^{n-2}f_*\omega_Y(E) = 0$ (**Main issue**.)

Local cohomological dimension

- Recall however that by $F_0 \subseteq E_0$ we have an injection

$$R^{n-2} f_* \omega_Y(E) \hookrightarrow \mathcal{E}xt^{n-1}_{\mathcal{O}_X}(\mathcal{O}_Z, \omega_X).$$

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Recovers:

Theorem (Ogus)

If $\operatorname{depth} \mathcal{O}_Z \geq 2$ *(e.g. Z normal), then* $\operatorname{lcd}_X(Z) \leq n-2$*.*

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If $\mathrm{depth}\, \mathcal{O}_Z \geq 3$*, then* $\mathrm{lcd}_X(Z) \leq n-3$.

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If $\mathrm{depth}\ \mathcal{O}_Z \geq 3$*, then* $\mathrm{lcd}_X(Z) \leq n-3$.

- This pattern stops here! There are examples with depth at least 4, but lcd $n-3$.

However...

Local cohomological dimension

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Corollary

If Z satisfies one of the following:

- *It is locally given by a monomial ideal. (Previously known: Lyubeznik, ...)*
- *It has quotient singularities.*

Then

$$\mathrm{lcd}_X(Z) = n - \mathrm{depth}\ \mathcal{O}_Z = \mathrm{pd}\ \mathcal{O}_Z.$$