

Quantum cohomology, isomonodromic deformations, and derived categories

Part I. Dubrovin's conjecture – an overview

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In memoriam



Boris A. Dubrovin (1950–2019)



Bumsig Kim (1968–2021)

Quantum cohomology

- ▶ Quantum cohomology
- ▶ Frobenius manifolds
- ▶ Quantum differential equations

Riemann–Hilbert–Birkhoff problems

- ▶ Monodromy data
- ▶ RHB inverse problem

Dubrovin's conjecture

- ▶ Exceptional collections
- ▶ Dubrovin's conjecture

Leitmotiv: To study relations between

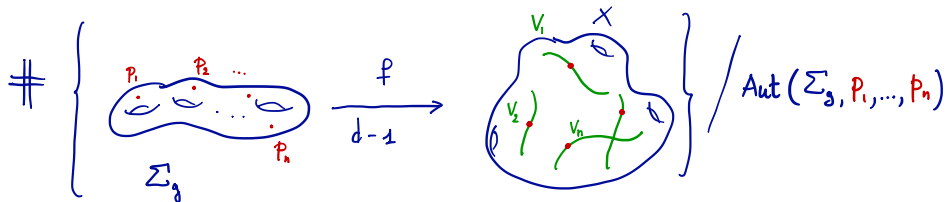
- ▶ the topology,
- ▶ the enumerative/symplectic geometry,
- ▶ the complex geometry,

of a smooth projective variety X over $\mathbb{C}$.

Such a study is developed via the analysis of meromorphic connections on $\mathbb{P}^1(\mathbb{C})$, and their **isomonodromic deformations**.

Gromov-Witten theory associates with any complex smooth projective variety X a family of Frobenius algebras, its quantum cohomology $QH^\bullet(X)$.

$QH^\bullet(X)$ is a deformation of $H^\bullet(X)$, via counting numbers of rational curves on X



$$\langle [V_1], [V_2], \dots, [V_n] \rangle_{g,n,d}^X := \int_{[\bar{\mathcal{M}}_{g,n}(X,d)]^{\text{virt}}} ev_1^*[V_1] ev_2^*[V_2] \dots ev_n^*[V_n]$$

$$\bar{\mathcal{M}}_{g,n}(X,d)$$

$$\downarrow ev_i \quad i=1, \dots, n$$

$$X$$

Gromov-Witten theory associates with any complex smooth projective variety X a family of Frobenius algebras, its quantum cohomology $QH^\bullet(X)$.

$QH^\bullet(X)$ is a deformation of $H^\bullet(X)$, via counting numbers of rational curves on X

Assume $H^{\text{odd}}(X, \mathbb{C}) \equiv 0$

$T_1, T_2, \dots, T_N$ homogeneous basis of $H^\bullet(X, \mathbb{C})$ [WLOG: $T_1 = 1$]
 $t^1, t^2, \dots, t^N$ dual coordinates

$$F_0^X(t) := \sum_{n=0}^{\infty} \sum_{d \in H_2(X, \mathbb{Z})} \sum_{\alpha_1, \dots, \alpha_n=1}^N \frac{t^{\alpha_1} \dots t^{\alpha_n}}{n!} \int \bigcup_{i=1}^n \text{ev}_i^* T_{\alpha_i} [\bar{M}_{0,n}(X, d)]^{\text{vir}}$$

ASSUMPTION: F_0^X has non-empty convergence domain Ω

TWO REMARKABLE PROPERTIES of F_0^X

1. QUASI-HOMOGENEITY

$$E := \sum_{\alpha=1}^N \left[(1 - q_\alpha) t^\alpha + r_\alpha \right] \frac{\partial}{\partial t^\alpha} \quad \text{Euler vector field on } \Omega$$

$$\text{where } T_\alpha \in H^{2q_\alpha}(X, \mathbb{C}) \quad \text{and} \quad \sum_{\alpha=1}^N r_\alpha T_\alpha = c_1(X)$$

We have

$$\mathcal{L}_E F_0^X = (3 - \dim_{\mathbb{C}} X) F_0^X + \text{quadratic}$$

2. Witten - Dijkgraaf - Verlinde - Verlinde equations

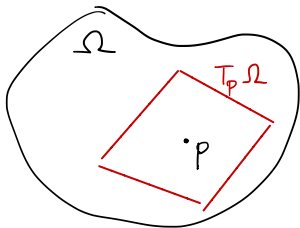
η Poincaré pairing:

$$\eta(\mathbb{Z}_1, \mathbb{Z}_2) := \int_X \mathbb{Z}_1 \cup \mathbb{Z}_2$$

$$\eta_{\alpha\beta} := \eta(T_\alpha, T_\beta)$$

$$\partial_{\alpha\beta\gamma}^3 F \cdot \gamma^{r\delta} \cdot \partial_{\delta\epsilon\eta}^3 F = \partial_{\eta\beta\gamma}^3 F \cdot \gamma^{r\delta} \cdot \partial_{\delta\epsilon\alpha}^3 F$$

We can define a new product $*_p$ on $T_p \Omega$



$$T_p \Omega \cong H^1(X, \mathbb{C})$$

$$\frac{\partial}{\partial t^\alpha} \mapsto T_\alpha$$

$$c_{\alpha\beta}^\lambda := \sum_\varepsilon \eta^{\varepsilon\lambda} \cdot \left. \frac{\partial^3 F_0^X}{\partial t^\alpha \partial t^\beta \partial t^\varepsilon} \right|_p$$

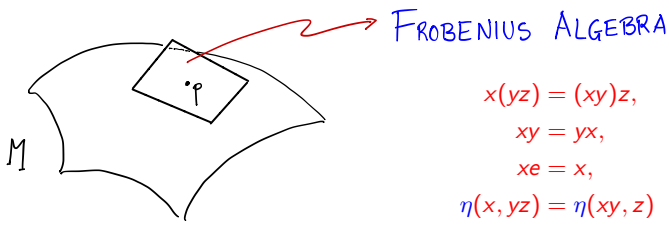
$$\frac{\partial}{\partial t^\alpha} *_p \frac{\partial}{\partial t^\beta} := \sum_\lambda c_{\alpha\beta}^\lambda \frac{\partial}{\partial t^\lambda}$$

QUANTUM PRODUCT
AT $p \in \Omega$

$*_p$ is COMMUTATIVE
ASSOCIATIVE
and UNITAL

$$\eta(\alpha *_p \beta, \gamma) = \eta(\beta, \alpha *_p \gamma)$$

Frobenius Manifolds are complex manifolds whose tangent spaces admit a *Frobenius algebra* structure.



$$x(yz) = (xy)z,$$

$$xy = yx,$$

$$xe = x,$$

$$\eta(x, yz) = \eta(xy, z) \quad \eta \text{ FLAT}$$

Examples coming from:

- Symplectic and Algebraic Geometry
- Singularity Theory

Milestones: Dubrovin, Hitchin, Kontsevich, Manin, Saito, Vafa, Witten, ...

Mirror Symmetry as isomorphism of Frobenius manifolds

$$QH^\bullet(X) \cong (V, f: V \rightarrow \mathbb{C})$$

There is a **local identification**

$$\left\{ \begin{array}{l} \text{semisimple points } t \text{ of} \\ \text{Frobenius } n\text{-manifold } M \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{def. param. of isomonodromic} \\ \text{families of differential systems } n \times n \\ \frac{dY(z, t)}{dz} = \left(U(t) + \frac{1}{z} V(t) \right) Y(z, t) \\ U \text{ diagonal, } V \text{ anti-symmetric} \end{array} \right\}$$

∇ LEVI-CIVITA connection of η
 $\hookrightarrow$ FLAT

$$\nabla_x^z Y = \nabla_x Y + z X * Y, \quad \begin{array}{l} X, Y \in \Gamma(T\Omega) \\ z \in \mathbb{C} \end{array}$$

Thm (DuBois)

∇^z is FLAT $\forall z \in \mathbb{C}$

There is a **local identification**

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∇ LEVI-CIVITA connection of η
 $\hookrightarrow$ FLAT

$$\hat{\nabla}_X Y = \nabla_X Y + z X * Y, \quad X, Y \in \Gamma(T\Omega)$$

$$\hat{\nabla}_{\frac{\partial}{\partial z}} Y = \frac{\partial}{\partial z} Y + E * Y - \underbrace{\frac{1}{z} \mu(Y)}_{\substack{\text{Euler field} \\ \frac{2-\dim X}{2} \cdot Y - \nabla_Y E}}$$

$$\begin{array}{ccc} \pi^* T\Omega & \longrightarrow & T\Omega \\ \downarrow \lrcorner & & \downarrow \\ \Omega^* \mathbb{C}^* & \xrightarrow{\pi} & \Omega \end{array}$$

Then

$\hat{\nabla}$ is FLAT

FLATNESS of $\hat{\nabla} \Rightarrow \exists f$ function on $\Omega \times \mathbb{C}^*$ s.t. $\hat{\nabla} df = 0$

$$Y := \text{grad}_\eta(f) \text{ satisfies } \begin{cases} \frac{\partial Y}{\partial t^\alpha} = \mathbb{L} \partial_\alpha * Y \\ \frac{\partial Y}{\partial z} = \left(u + \frac{1}{2}\mu\right) Y \end{cases}$$

ASSUME that Ω is a
SEHISIMPLE FROBENIUS MANIFOLDS

$\exists$ idempotent vector
fields $\pi_1, \dots, \pi_n \in \Gamma(T\Omega)$

$$\pi_i * \pi_j = \delta_{ij} \pi_i \quad i, j = 1, \dots, n$$

$$\eta(\pi_i, \pi_j) = 0 \quad \text{if } i \neq j$$

Thm (Dubrovin)

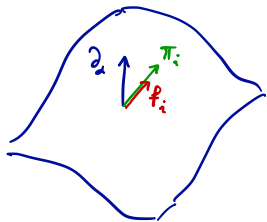
1. We have $[\pi_i, \pi_j] = 0$.

2. The eigenvalues $u_1, \dots, u_n$ of the operator $u = E*(-)$ satisfy $\frac{\partial}{\partial u_i} \equiv \pi_i$.

FLATNESS of $\hat{\nabla} \Rightarrow \exists f$ function on $\Omega \times \mathbb{C}^*$ s.t. $\hat{\nabla} df = 0$

$$Y := \text{grad}_\eta(f) \text{ satisfies } \begin{cases} \frac{\partial Y}{\partial t^\alpha} = \frac{1}{2} \partial_\alpha * Y \\ \frac{\partial Y}{\partial z} = \left(u + \frac{1}{2}\mu\right) Y \end{cases}$$

ASSUME that Ω is a
SEMI-SIMPLE FROBENIUS MANIFOLDS



$$f_i := \frac{\pi_i}{\eta(\pi_i, \pi_i)^{\frac{1}{2}}} \quad i=1, \dots, n$$

$\exists$ idempotent vector
fields $\pi_1, \dots, \pi_n \in \Gamma(T\Omega)$

$$\pi_i * \pi_j = \delta_{ij} \pi_i \quad i, j=1, \dots, n$$

$$\eta(\pi_i, \pi_j) = 0 \quad \text{if } i \neq j$$

We can rewrite the $\hat{\nabla}$ -coflatness
system in the orthonormalized idempotent
frame

$$\partial_\alpha = \sum_{i=1}^n \Psi_{i\alpha} f_i$$

To each point of $QH^\bullet(X)$ there is an attached differential equation

$$\frac{dY}{dz} = \left(U(t) + \frac{1}{z} V(t) \right) Y, \quad z \in \mathbb{C}^*, \quad t \in QH^\bullet(X).$$

Its solutions are multivalued, and they manifest a Stokes phenomenon.

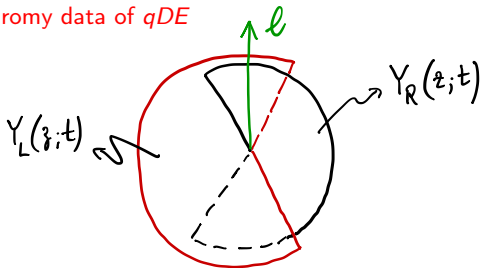
→ Monodromy data of qDE

FUCHSIAN SINGULARITY $z=0$

IRREGULAR SINGULARITY $z=\infty$

$Y_0(z;t) \rightsquigarrow$ solution at $z=0$

$Y_F(z;t) \rightsquigarrow$ formal solution with
exponential expansion
at $z=\infty$



$$Y_R(z;t) = Y_0(z;t) \cdot C(t),$$

$$Y_L(z;t) = Y_R(z;t) \cdot S(t)$$

► Isomonodromic property: monodromy data are locally constant wrt t .

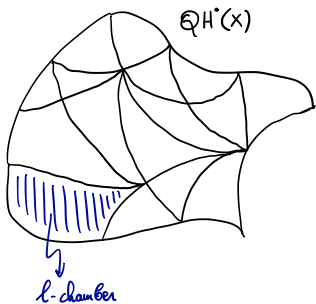
FUCHSIAN SINGULARITY at $z=0$

FUNDAMENTAL SOLUTIONS in
LEVEL FORM

$$Y_0(z, t) = \Psi(t) \cdot \Phi(z, t) \cdot z^{\mu} z^R$$

$$\Phi(z, t) = 1 + \sum_{k=1}^{\infty} \Phi_k(t) z^k$$

$$\Phi(-z, t)^T \eta \Phi(z, t) = \eta$$



IRREGULAR SINGULARITY at $z=\infty$

GENUINE SOLUTIONS $Y_{L/R}$

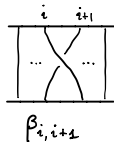
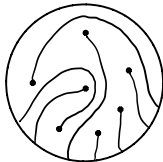
$$Y_{L/R}(z, t) \sim \mathbb{U}(z, t) e^{\pm U(t)}$$

$$\mathbb{U}(z, t) = 1 + \sum_{n=1}^{\infty} \mathbb{U}_n(t) z^{-n}$$

$$\mathbb{U}(-z, t)^T \mathbb{U}(z, t) = 1.$$

WALL-CROSSING PHENOMENON:

BRAID GROUP B_n - ACTION ON
BOTH (S, c) AND EXC. COLL.



Several non-canonical choices of normalizations:

1. Action of $\tilde{G}_n$ (choice of ORDERING of CANONICAL COORDINATES)

$$S \mapsto P S P^{-1} \quad C \mapsto C P^{-1}$$

2. Action of $(\mathbb{Z}/2\mathbb{Z})^n$ (choice of signs of normalized idempotents)

$$S \mapsto J S J^{-1} \quad C \mapsto C J^{-1}$$

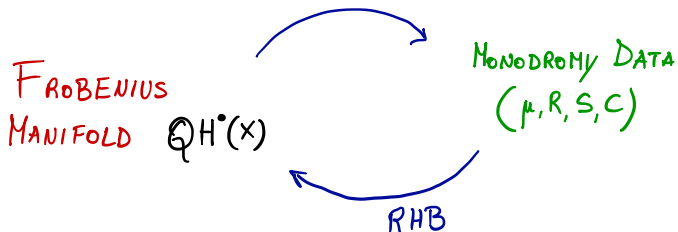
3. Action of $\mathbb{Z}$ (choice of slope of l)

$$C \mapsto M_0^k C, \quad M_0 := \exp(2\pi\sqrt{-1}\mu) \exp(2\pi\sqrt{-1}R)$$

4. Action of $\mathcal{C}_0(\mu, R)$ (choice of the Levelt normal form at $z=0$)

$$C \mapsto G C$$

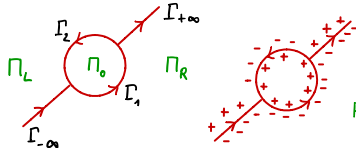
RHB inverse problem



The monodromy data define
a "system of coordinates" in the
space of quasi-homogeneous
solutions of WDVV equations

RHB inverse problem

Let $u_0 \in \mathbb{C}^n \setminus \Delta$
inside an l -chamber



$$H: \Gamma \rightarrow GL(n, \mathbb{C})$$

$$H(z; u_0) = \begin{cases} e^{zU} S e^{-zU} & z \in \Gamma_{+\infty} \\ e^{zU} S^T e^{-zU} & z \in \Gamma_{-\infty} \\ e^{zU} C^{-1} z^{-R} z^{-P} & z \in \Gamma_1 \\ S^{-1} C^{-1} z^{-R} z^{-P} & z \in \Gamma_L \end{cases}$$

RHB Problem Find an analytic

$G: \mathbb{C} \setminus \Gamma \rightarrow M_n(\mathbb{C})$ such that

- $G|_{\Pi_\nu} \in \mathcal{C}^0(\overline{\Pi_\nu})$, $\nu=0, L, R$
- $\exists$ non-tangential limits
 $G_\pm: \Gamma^\circ \rightarrow M_n(\mathbb{C})$ and $G_+ = G_- H$
- $G(z) \rightarrow I_n$ as $z \rightarrow \infty$

Then (Malgrange) If the RHB problem is solvable at $u_0 \in \mathbb{C}^n \setminus \Delta$, then it is solvable in a neighborhood of u_0 . The solution is holomorphic on u .

If $G(z, u) = G_0(u) + z G_1(u) + z^2 G_2(u) + z^3 G_3(u) + O(z^4)$, $z \rightarrow 0$ in Π_0
then

$$t^\alpha(u) := \eta^{\alpha\beta} \sum_{i=1}^n G_{0,i\beta}(u) G_{1,i1}(u), \quad \alpha = 1, \dots, n,$$

$$F(u) := \frac{1}{2} \left[t^\alpha(u) t^\beta(u) \sum_{i=1}^n G_{0,i\alpha}(u) G_{1,i\beta}(u) - \sum_{i=1}^n G_{1,i1}(u) G_{2,i1}(u) G_{0,i1}(u) G_{3,i1}(u) \right].$$

This is a solution
of WDVV

A result on the convergence of $F_o^X(\pm)$

1. C. Sabbah recently proved an extension of Malgrange Thm, dealing with the case $u_o = (u_o^1, \dots, u_o^n)$, $u_o^i = u_o^j$, $i \neq j$.

2. Thm (E.) Assume that

(i) the series $\sum_{d \in H_2(X, \mathbb{Z})} \sum_{\alpha_1, \dots, \alpha_n} \int_{\bar{U}_{o,n}(X, d)} \bigcup_{i=1}^n \text{ev}_i^* T_{\alpha_i}$ converges $\forall n \geq 3$;

(ii) the $\mathbb{C}$ -Frobenius algebra $(H^*(X, \mathbb{C}), *_o)$ is semisimple;

Then $F_o^X(\pm)$ is convergent on a non-empty Ω .

Rk: if X is Fano, then (i) holds true.

Dubrovin Conjecture (as refined in [CDG18])

Let X be a Fano s.p.v./ $\mathbb{C}$, with $h^{p,q}(X) = 0$ unless $p = q$.

[DC1]. $QH^\bullet(X)$ is a semisimple iff $D^b(X)$ admits a full exceptional collection.

Reminder A, B objects in $\mathcal{D}^b(X)$

$$\begin{array}{ccc} & \alpha \swarrow C & \\ A & \xrightarrow{f} & B \\ & \nwarrow \alpha & \end{array}$$

α quasi-isomorphism

$(E_1, \dots, E_n)$ in $\mathcal{D}^b(X)$

$$\mathrm{Hom}^\bullet(E_i, E_i) = \mathbb{C} \quad i=1, \dots, n$$

$$\mathrm{Hom}^\bullet(E_j, E_i) = 0 \quad j > i$$

FULL if
it generates
 $\mathcal{D}^b(X)$

$$\begin{array}{c} E_1 \dots E_i \quad E_{i+1} \dots E_n \\ \hline \begin{array}{c} \vdots \\ \vdots \end{array} \quad \begin{array}{c} \diagdown \quad \diagup \\ \vdots \end{array} \quad \begin{array}{c} \vdots \\ \vdots \end{array} \\ \hline E_i \dots L_{E_i} E_{i+1} \quad E_i \dots E_n \end{array}$$

$$\mathrm{Hom}^\bullet(E_i, E_{i+1}) \otimes E_i \rightarrow E_{i+1} \rightarrow L_{E_i} E_{i+1}$$

Action of B_n

Prototypical example (Beilinson)

$$X = \mathbb{P}^n,$$

$$(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \dots, \mathcal{O}(n-1))$$

is a full exceptional collection.

Dubrovin Conjecture (as refined in [CDG18])

Let X be a Fano s.p.v./ $\mathbb{C}$, with $h^{p,q}(X) = 0$ unless $p = q$.

[DC1]. $QH^\bullet(X)$ is a semisimple iff $D^b(X)$ admits a full exceptional collection.

[DC2]. Fix $\ell = \{\arg z = \phi\}$, $\phi \in \mathbb{R}$. There is a map

$$\{\ell\text{-chambers of } QH^\bullet(X)\} \rightarrow \{\text{full exc. coll. in } D^b(X)\}, \quad \Omega \mapsto \mathfrak{E}_\Omega,$$

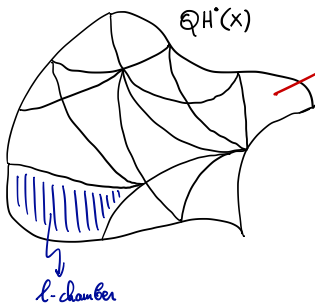
such that

1. the Stokes matrix in a chamber Ω equals the **inverse of the Gram matrix** of $\chi: K(X)_\mathbb{C} \times K(X)_\mathbb{C} \rightarrow \mathbb{Z}$ w.r.t. the basis $[\mathfrak{E}_\Omega]$;
2. the central connection matrix C in a chamber Ω equals the matrix associated with the morphism

$$\mathcal{D}_X^-: K_0(X) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow H^\bullet(X, \mathbb{C}): E \mapsto \frac{i^{\bar{d}}}{(2\pi)^{\frac{d}{2}}} \hat{\Gamma}_X^- \cup e^{-\pi i c_1(X)} \cup \text{Ch}(E),$$

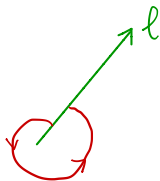
$$\text{Ch}(E) := \sum_j e^{2\pi\sqrt{-1}\alpha_j}, \quad \alpha'_j \text{ s Chern roots of } E,$$

$$\hat{\Gamma}_X^- := \prod_\ell \Gamma(1 - \varepsilon_\ell), \quad \varepsilon'_\ell \text{ s Chern roots of } TX,$$



$(E_1, \dots, E_n)$

S, C determined by the exceptional collection



If we rotate l counterclockwise by 2π , the l -chamber decomposition does NOT change. But the exceptional collection does!

Given $(E_1, \dots, E_n)$ we construct an infinite family, a **HELIX**, via iteration of left/right mutations.

$\dots E_{-1}, E_0, \underline{E_1, E_2, \dots, E_n}, E_{n+1}, E_{n+2}, \dots$

foundation of the helix

If $(E_1, \dots, E_n)$ is associated with l then $(E_{-n+1}, \dots, E_0)$ is associated with $e^{2\pi\sqrt{-1}} l$. **HELIX STRUCTURE**

$\dots, S(E_1), S(E_2), \dots, S(E_n), E_1, \dots, E_n, \dots$

↳ **SERRE FUNCTOR**

Historical evolutions of the statement of [DC]:

- ▶ Fano assumption: it was originally claimed that

$$X \text{ Fano} \iff QH^\bullet(X) \text{ semisimple}$$

Both $\Rightarrow$ and $\Leftarrow$ are **false**.

" $\Rightarrow$ " $\exists$ 47 families of Fano threefolds with $b_3(X) > 0$.
Hence $QH^\bullet(X)$ is not semisimple [Hertling-Mumford-Teleman]

" $\Leftarrow$ " Semisimplicity is preserved by blow-ups [Bayer]

Historical evolutions of the statement of [DC]:

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Both $\Rightarrow$ and $\Leftarrow$ are **false**.

- Originally, it was claimed $S_{ij} = \chi(E'_i, E'_j)$ for some $\mathfrak{E}' = (E'_1, \dots, E'_n)$.

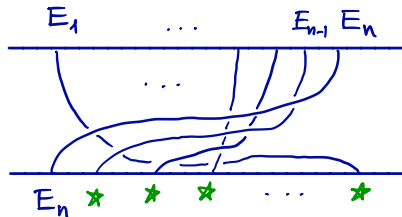
There is no contradiction:

$$\mathfrak{E}' = {}^\vee(\mathfrak{E}^*) \quad \text{or} \quad \mathfrak{E}' = (\mathfrak{E}^*)^\vee$$

$$\mathfrak{E} = (E_1, \dots, E_n)$$

$$\mathfrak{E}^* := (E_n^*, \dots, E_1^*)$$

$$\mathfrak{E}^\vee :=$$



Historical evolutions of the statement of [DC]:

- ▶ Fano assumption: it was originally claimed that

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There is no contradiction:

$$\mathfrak{E}' = {}^\vee(\mathfrak{E}^*) \quad \text{or} \quad \mathfrak{E}' = (\mathfrak{E}^*)^\vee$$

- ▶ Originally, C was conjectured to be of the form

$$C = C' C''$$

$C' \rightarrow$ components of $\text{ch}(E_i)$,

$C'' \rightarrow$ operator $C'' \in \text{End}(H^\bullet(X))$ s.t. $C''(c_1(X) \cup \alpha) = c_1(X) \cup C''(\alpha)$

The central connection matrix is computed w.r.t. the
topological-enumerative solution

$$\begin{array}{ccc}
 & & Y_0(z, t) = \Psi(t) \cdot \Theta_{\text{top}}(z, t) z^\mu z^{c_1(X) \cup}, \\
 \mathcal{L}_1 & & \\
 \downarrow T_{s_1}^* C & & (\Theta_{\text{top}})_{\beta}^{\alpha} := \delta_{\beta}^{\alpha} + \sum_{\lambda} \left\langle \left\langle \frac{z T_{\beta}}{1 - z \psi_1}, T_{\lambda} \right\rangle \right\rangle_0 \eta^{\lambda \alpha}, \\
 \overline{\mathcal{M}}_{0,n}(X, \beta) & & \psi_1 := c_1(\mathcal{L}_1)
 \end{array}$$

In 2014, S. Galkin, V. Golyshev, and H. Iritani formulated I - conjecture II
as a refinement of [DC2].2.

Different choice of Levelt normal form

→ Difference of exceptional collections

$\mathcal{L}$
 $\mathcal{E}^*$

Some further comments

1. $[DC2].2 \Rightarrow [DC2].1$

$$(e^{-\pi\sqrt{-1}\mu} C)^T \eta e^{\sqrt{-1}\pi R} C = S^{-1}$$

"ISOMONODROMIC INCARNATION"
f GHRR

2. Physical Motivations: tt^* -geometry

- qDE and tt^* -isomonodromy equations have the same Stokes phenomenon (Its, Guest, Lin)
- Zaslav conjecture

3. Compatibility with HMS

$$\begin{array}{ccc} \mathcal{Fuk}(X) & & \mathcal{D}^b(X) \\ & \nwarrow \nearrow & \\ \mathcal{DFuk}(V, f) & & \mathcal{D}_{sing}(V, f) \end{array}$$