Nilpotent Endomorphisms of Expansive Group Actions

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Plan

Nilrigidity

Main theorem and unpacking it

Tiered dynamical systems

Motivation for introducing tiered systems

Questions and assorted facts

Definitions: pointed dynamical systems

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- When we have both, and the actions commute, we write this as (G, X, f).

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- We say a family of ($\mathbb{N}$-)PDS $\mathcal{F}$ is *nilrigid* if for all $(X, f) \in \mathcal{F}$, AN implies UAN. We also say (X, f) is nilrigid if $\{(X, f)\}$ is.

One-dimensional cellular automata

- The *pointed (one-dimensional) full shift* is $X = A^{\mathbb{Z}}$, with zero $0 = 0^{\mathbb{Z}}$ for some $0 \in A$, and the action is by translations, $(1 \cdot x)_i = \sigma(x)_i = x_{i+1}$. A is the *alphabet*.

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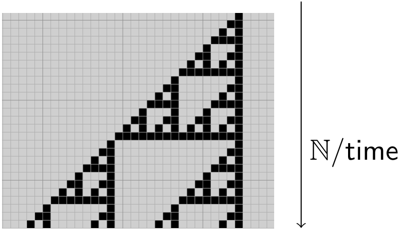
- $A = \mathbb{Z}/2\mathbb{Z}$, $f(x)_i = x_i + x_{i+1}$ gives a *XOR CA*



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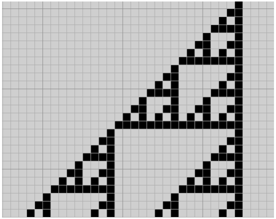

$\downarrow$ $\mathbb{N}$/time

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- XOR is not asymptotically nilpotent.

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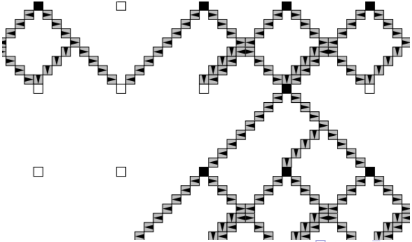
- There exist non-AN CA where $f^n \to 0$ uniformly in the Besicovitch topology (= spatial density) [Törmä,'14],



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- Nilpotency of cellular automata is undecidable (Σ^0_1-complete). Proved by [Aanderaan-Lewis,'74]

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- Basis of many undecidability results; undecidability of conjugacy of cellular automata was proved by a reduction from nilpotency [Jalonen-Kari, '20].

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Cellular automata (on one-dimensional full shifts) are nilrigid.

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Cellular automata (on one-dimensional full shifts) are nilrigid.

- Proof is specific to full shifts (or transitive SFTs) on $\mathbb{Z}$.

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- A *cellular automaton (CA)* $f : X \to X$ is a shift-commuting continuous map, by default on a full shift. When there's a zero, it is preserved.
- $X = A^G$ is an SFT, and its cellular automata are easy to enumerate: if $B \Subset G$ is finite and $F : A^B \to A$ arbitrary, then $f(x)_g = F(gx|_B)$ defines a CA.

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- It is easy to conclude the result for all abelian groups G.
- Strong irreducibility implies dense homoclinic points in the pointed setting.

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$$
\begin{tikzcd}[column sep=small]
 & \text{all f.g. groups} \arrow[d, dash] & \\
 & \text{sofic} \arrow[dl, dash] \arrow[dr, dash] & \\
\text{amenable} \arrow[d, dash] & & \text{LEF} \ni S_\infty \rtimes \mathbb{Z} \arrow[dd, dash] \\
\text{elem. amenable} \ni S_\infty \rtimes \mathbb{Z} \arrow[d, dash] & & \\
\text{solvable} \ni \mathbb{Z}_2 \wr \mathbb{Z} \arrow[dr, dash] & & \text{residually finite} \ni \mathbb{Z}_2 \wr \mathbb{Z}, F_2 \arrow[dl, dash] \\
 & \text{polycyclic} \arrow[d, dash] & \\
 & \text{nilpotent} \ni H_3 \arrow[d, dash] & \\
 & \text{abelian} \ni \mathbb{Z}, \mathbb{Z}^2 &
\end{tikzcd}
$$

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Let G be locally (residually finite and poly-(locally virtually abelian)). Let $(G, X, 0)$ be an expansive zero-gluing pointed dynamical system with dense homoclinic points. Then endomorphisms of $(G, X, 0)$ are nilrigid.

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Expansivity and endomorphisms

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- $(G, X, 0)$ is *expansive* if there exists $\epsilon > 0$ such that

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- an *endomorphism* is a continuous map $f : X \to X$ s.t. $\forall g \in G, x \in X : gf(x) = f(gx)$. Endomorphisms of subshifts are cellular automata.

Zero-gluing

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- Shadowing/POTP implies zero-gluing, but not vice versa.

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That group pic again

all f.g. groups

sofic

amenable

LEF $\ni S_\infty \rtimes \mathbb{Z}$

elem. amenable $\ni S_\infty \rtimes \mathbb{Z}$

solvable $\ni \mathbb{Z}_2 \wr \mathbb{Z}$

residually finite $\ni \mathbb{Z}_2 \wr \mathbb{Z}, F_2$

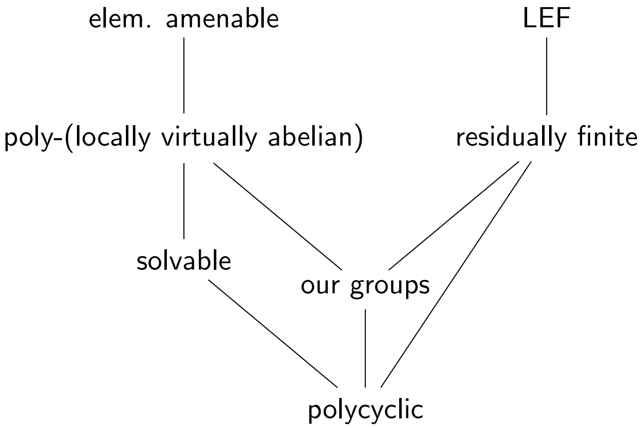
polycyclic

nilpotent $\ni H_3$

abelian $\ni \mathbb{Z}, \mathbb{Z}^2$

Where are our groups?

Restricting to f.g:



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  - several (classes of) groups which are not virtually solvable.

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- The $\mathbb{Z}$-action by the hyperbolic toral automorphism $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is expansive, shadowing and has dense homoclinic points. Ditto for the x2 map on the 2-solenoid $\lim S^1$.

Plan

Rough sketch of our paper

Our paper uses the periodization idea from [S.,'12], but $\mathbb{Z}^{d-1} \times \mathbb{Z}$ is replaced by a group extension K-by-H. The induction step requires proving nilrigidity in higher generality.

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4. This specializes to our main theorem for single tier systems.

Tiered dynamical systems

Definition

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We denote $G_{t_0} = G_0$, and call it the *base group* of $\mathcal{X}$.

Evolutions of tiered dynamical systems

Definition

An *evolution map*, or simply *evolution*, of a TTDS $\mathcal{X}$ is a continuous function $f : X \to X$ such that for all $t \leq t' \in \mathcal{D}$ and $x \in X_t$, the condition $f(x) \in X_{t'}$ implies $f(g \cdot x) = g \cdot f(x)$ for all $g \in G_{t'}$.

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We say that a tier $t \in \mathcal{D}$ is *f-stabilizing*, if there exists $s(t) \geq t$ such that $f^k(X_t) \subseteq X_{s(t)}$ for all $k \in \mathbb{N}$. The TTDS $\mathcal{X}$ is f-stabilizing if each of its tiers is. The quadruple $(X, (G_t, X_t)_{t \in \mathcal{D}}, (\phi_t^{t'})_{t \leq t'}, f)$, which we may denote by $(\mathcal{X}, f)$, is called an *evolution of a tiered (topological) dynamical system*, or ETTDS for short.

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  - Clearly f is AN also on (H, Y_p). Therefore $f^n(Y_p) = \{0\}$ for some n!

More on the proof in the abelian case 2

- After the previous observation, the proof splits into two cases: if $K \lhd G$ is a normal subgroup and $f : A^G \to A^G$ a CA, say *information cannot spread arbitrarily far from* K if

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- If information can spread arbitrarily far from one of these groups, repeatedly apply the argument from the previous slide to contradict AN: "shoot and kill by periodization".

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What is different from the abelian case?

- Proof idea: Suppose a large family of dynamical systems has nilrigid endomorphisms on H and K.
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- These are just the Fix and Fin-constructions, respectively.

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- The Heisenberg group is $H_3 = \langle a, b, c \mid c = [a, b], [a, c], [b, c] \rangle$.

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- If information cannot spread arbitrarily far from $\langle c \rangle$, then $\forall t : \exists t' : \forall n : f^n(X_t) \subset X_{t'}$, i.e. we have f-stabilization.
- Then f is a stabilizing ETTDS, apply nilrigidity of $\mathbb{Z}$-ETTDS to obtain the result.

Nilrigidity on the Heisenberg group (cont'd)

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- f-stabilization is trivial: $f^n(X_t) \subset X_t$ by shift-commutation, and f is an AN evolution.
- Mimic argument from the abelian case using nilrigidity of $\mathbb{Z}^2$-ETTDS. This contradicts AN for f!

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Are CA nilrigid on full shifts on all groups? On SFTs with dense homoclinics (e.g. strongly irreducible SFTs)?

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Are cellular automata nilrigid on all $\mathbb{Z}^2$-SFTs?

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Theorem (S.,'19)

Let T be the infinite k-regular tree and G its automorphism group. Then endomorphisms of (G, A^T) are nilrigid.

The End

Thank you for listening!