Rotated Odometers

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Rotated odometers

Rotated odometers are a class of infinite interval exchange transformations (IET).

Motivations for study:

- Model first return maps of flows on translation surfaces of infinite type,

- Can be considered as perturbation of the von Neumann-Kakutani map (standard dyadic odometer).

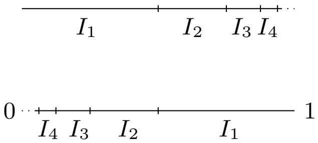
Methods: Bratteli diagrams.

Results: Some results towards the classification up to an isomorphism (measure-theoretical and continuous factors).

The von Neumann-Kakutani map

$$A(x) = x - (1 - 3 \cdot 2^{1-n}) \qquad \text{if } x \in [1 - 2^{1-n}, 1 - 2^{-n}),\ n \geq 1,$$

is an infinite IET.



There exists a Cantor set $C = \{0,1\}^{\mathbb{Z}}$ and a map $\iota : I \to C$ such that

1. $\iota(I)$ is dense in C,
2. $\iota \circ A(x) = \iota(x) + 1$, where addition is with carry over to the right.

So (I, A) is measurably isomorphic to the standard dyadic odometer.

Let $q \geq 2$ and let π be a permutation of q symbols.

Divide $I = [0, 1)$ into q intervals, then π induces a finite IET $R_\pi : I \to I$.

A **rotated odometer** is an infinite IET

$$F_\pi = A \circ R_\pi : I \to I$$

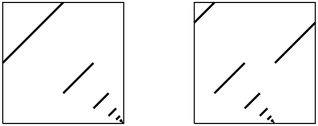


Figure: The von Neumann-Kakutani map A and a rotated odometer $F_{(012)}$.

Problem

Let $q \geq 2$, let π be any permutation of q symbols.

Let $R_\pi : I \to I$ be the corresponding finite IET (on intervals of equal length), and let

$$F_\pi = A \circ R_\pi : I \to I$$

be a rotated odometer.

Study the dynamics and ergodic properties of such systems.

But first, what is the motivation for studying rotated odometers?

Rotated odometers model first return maps of flows on infinite-type translation surfaces.

Theorem (Bruin and Lukina 2021)

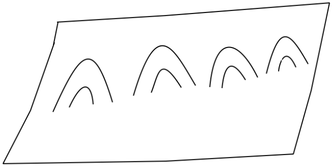
Let $q \geq 2$, and π be a permutation of q intervals.

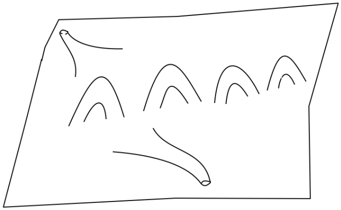
Let λ be Lebesgue measure on I.

Then there exists a translation surface of infinite type, which is topologically a Loch-Ness monster, such that (I, F_π, λ) is measurably isomorphic to the first return map of a parallel flow on this surface.

Note: more precise formulations will follow.

Surfaces which appear in the previous theorem: a Loch-Ness monster and a Loch-Ness monster with whiskers.





More details at the end of the talk, if time permits.

Consider a rotated odometer (I, F_π) as a perturbation of the standard von Neumann-Kakutani map (I, A).

What properties of (I, A) are preserved under such perturbation?

Theorem (Bruin and Lukina 2021)

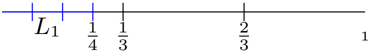
1. Minimality is not preserved, and (I, F_π) may have periodic points; however, there is always an aperiodic subsystem (I_{np}, F_π) with unique minimal set (I_{min}, F_π).
2. Often (I_{min}, F_π) and even (I_{np}, F_π) have the dyadic odometer as a maximal equicontinuous factor.
3. However, there are examples where neither (I_{min}, F_π) nor (I_{np}, F_π) have the dyadic odometers as equicontinuous factors, moreover, equicontinuous factors for (I_{min}, F_π) are (I_{np}, F_π) are different.

Dynamics of rotated odometers - method of coding partitions

(I, F_π) rotated odometer, where π is a permutation of q symbols

sections $L_k = [0, 2^{-kN})$, where $N = \min\{n \mid 2^{-n} < q^{-1}\}$,

return maps $F_k : L_k \to L_k$, with $L_0 = I$.



$\mathcal{P}_{q,k}$ is a partition of I into $q2^{kN}$ intervals

$\mathcal{P}_{q,k}$ induces a partition of L_k into q intervals

Coding

The partition of L_{k-1} into q intervals from $\mathcal{P}_{q,k-1}$ is the coding partition for the orbits of q intervals in L_k under F_{k-1}.

Using this approach, we can obtain the following:

- $I = I_{per} \cup I_{np}$, where I_{per} is a possibly infinite union of intervals of periodic points, I_{np} consists of non-periodic points

- for each $k \geq 1$, F_k is a composition of a permutation R_k of q intervals, and a scaled von Neumann-Kakutani map.

- $0 \in I_{np}$, and the orbit of every non-periodic point accumulates at 0.

Lemma

The aperiodic subsystem (I_{np}, F_π) has a unique minimal set.

We now concentrate on the study of (I_{np}, F_π).

Cantor set representation

Recall that the odometer A is continuous on $[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n})$, $n \geq 0$.

The rotated odometer F_π is also continuous on subintervals in I.

Lemma

The set C of forward and backward orbits of the points $\{R_\pi^{-1}(1 - \frac{1}{2n}) \mid n \geq 0\}$ under F_π is dense in I_{np}.

Adding limit points to I_{np} we obtain

$$I_{np}^* = I_{np} \cup \{x^- \mid x \in C\} \cup \{1\},$$

which is a Cantor set with order topology.

The extension $F_\pi : I_{np}^* \to I_{np}^*$ is a homeomorphism.

To summarize,

Theorem

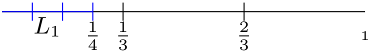
There exists a Cantor set I_{np}^*, and an injective map $\iota : I_{np} \to I_{np}^*$, such that:

1. The image $\iota(I_{np})$ is dense in I_{np}^*.
2. The IET F_π extends to a homeomorphism $F_\pi : I_{np}^* \to I_{np}^*$ in an equivariant manner, i.e., $\iota \circ F_\pi = F_\pi \circ \iota$.

Remark: The forward orbit of $0 \in I_{np}$ is joined with the backward orbit of one of the added points. So the orbit of 0 under F_π in I_{np}^* is two-sided.

S-**adic systems**:

Recall that we code orbits of q sets in the partition $\mathcal{P}_{q,k}$ of L_k under F_{k-1} using the partition $\mathcal{P}_{q,k-1}$ of L_{k-1} into q sets.



Recording the set of $\mathcal{P}_{q,k-1}$ visited by the orbit of an interval in $\mathcal{P}_{q,k}$ under F_{k-1}, we obtain substitutions

$$\chi_k(i),\ 0 \leq i \leq q-1 \text{ with alphabet } \mathcal{A} = \{0, 1, \ldots, q-1\}.$$

Lemma

The sequence $\{\chi_k\}_{k\geq 1}$ is eventually periodic.

Bratteli diagrams

We associate to the sequence $\{\chi_k\}_{k\geq 1}$ a Bratteli diagram:

- $V_0 = \{v_0\}$, $V_k = \{0, 1, \ldots, q-1\}$ for $k \geq 1$,

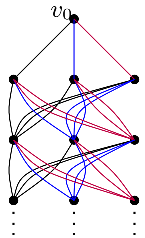
- in the set E_k of edges between $i \in V_{k+1}$ and $j \in V_k$, the number of edges from j to i is equal to the number of occurrences of j in $\chi_k(i)$,

- edges incoming to $i \in V_{k+1}$ inherit the order from $\chi_k(i)$.

The total order on edges incoming to each vertex induces a reverse lexicographic order on the space of infinite paths X.

Lemma: The diagram (V, E) has a unique minimal and unique maximal paths.

Consequence: (V, E) admits a Vershik system (X, τ).



A subdiagram (V', E') of (V, E) consists of a subset of vertices V' of V, and edges in E such that both starting and ending vertices of the edge are in V'; there is a corresponding subset X' of paths in X, and τ determines a Vershik map τ' on X'.

Theorem

There exists a subdiagram (V', E') of (V, E) with associated Bratteli-Vershik system (X', τ') such that there is a homeomorphism

$$\psi : I^*_{np} \to X'$$

and $\psi \circ F_\pi = \tau' \circ \psi$.

There is also a subdiagram $(\widehat{V}, \widehat{E})$ with Bratteli-Vershik system $(\widehat{X}, \widehat{\tau})$ conjugate under ψ to the minimal subsystem (I^*_{min}, F_π) of (I^*_{np}, F_π).

Ergodic measures:

Consider the systems (I, F_π), (I^*_{np}, F_π) and (I^*_{min}, F_π).

Theorem

For the rotated odometer (I, F_π), the Lebesgue measure λ is ergodic if and only if (I, F_π) has no periodic points.

The minimal subsystem (I^*_{min}, F_π) is uniquely ergodic (**Durand 2000**).

Example: Rotation by $1/3$ corresponds to $\pi = (012)$, by $2/3$ to $\pi = (021)$.

In either case, the rotated odometer (I, F_π) has no periodic points, so $I = I_{np}$; also $(I^*_{np}, F_\pi) = (I^*_{min}, F_\pi)$.

Thus I^*_{np} has unique ergodic measure which corresponds to the Lebesgue measure on I.

Non-minimal (I^*_{np}, F_π): use the results of **Bezuglyi, Kwiatkowski, Medynets, Solomyak 2010**.

The number of ergodic measures corresponds to eigenvalues greater than 1 with non-negative left eigenvectors in the Frobenius form of the substitution matrix for the periodic part of the sequence $\{\chi_k\}_{k\geq 1}$.

It follows that there may be at most q invariant ergodic measures.

So far in examples we have seen at most 2 invariant ergodic measures on (I^*_{np}, F_π).

Example: Let $q = 5$ and $\pi = (01234)$, so R_π is a rotation by $1/5$.

Then $\{\chi_k\}_{k\geq 1}$ is constant.

There are no periodic points, so Lebesgue measure is ergodic.

The substitution matrix in the Frobenius form (after renaming symbols)

$$B = \left(\begin{array}{cc|cc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 4 & 4 & 8 & 8 & 8 \end{array}\right) \text{ with } F_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ F_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ F_3 = (8).$$

There are two ergodic measures, μ_1 supported on I^*_{min}, and μ_2 supported on I^*_{np} and corresponding to the Lebesgue measure.

Measure-theoretical and continuous rotational factors for (I^*_{min}, F_π)

We study the spectrum (eigenvalues) of the Koopman operator for (I^*_{min}, F_π). Recall that:

- All eigenvalues of the Koopman operator are on the unit circle, i.e. of the form $\zeta = e^{2\pi i\theta}$, $\theta \in \mathbb{R}$.

- ζ is a measurable eigenvalue if the corresponding eigenfunction is measurable.

- ζ is a continuous eigenvalue if the corresponding eigenfunction is continuous.

For substitutions, eigenfunctions are always continuous, so measurable and continuous factors coincide (**Host 1986**).

Theorem (Bruin and Lukina 2021)

Let (I^*_{min}, F_π) be the minimal subset of (I^*_{np}, F_π). Then:

1. There exist infinitely many $q \geq 3$ and permutations π of q symbols, such that the minimal system (I^*_{min}, F_π) has a dyadic odometer as a factor.
2. There exist infinitely many $q \geq 3$ and permutations π of q symbols, such that the minimal system (I^*_{min}, F_π) does not factor to a dyadic odometer, but is not weakly mixing.

Part 1: Let $q = 5$ and $\pi = (01234)$, then $\{\chi_k\}_{k \geq 1}$ is constant.

The minimal subdiagram has 0 and 3 as vertices.

The substitution matrices for the full diagram and the minimal subdiagram are

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 4 & 8 & 8 & 4 & 8 \end{pmatrix} \text{ with } B_{min} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

B_{min} has eigenvalues 0 and 2, and it follows that $e^{2\pi i/2^m}$ is an eigenvalue for any $m \geq 1$.

It follows that the dyadic odometer is a factor of (I^*_{min}, F_π); in fact the factor map is a conjugacy.

Part 2: Let $q = 5$ and let $\pi = (02431)$, then $\{\chi_k\}_{k\geq 1}$ is constant.

(I, F_π) has no periodic points, so the Lebesgue measure is ergodic.

The substitution matrices for the full diagram and the minimal subdiagram are

$$B = \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 5 & 3 & 8 & 5 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \text{ with } B_{min} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

B_{min} has eigenvalues $2 \pm \sqrt{5}$, -1 of multiplicity 2.

Only $2 \pm \sqrt{5}$ are important, since the initial vector $h^{(1)}$ of heights for the Bratteli-Vershik diagram is decomposed over eigenvectors of $2 \pm \sqrt{5}$.

Using **Ferenczi, Mauduit, Nogueira 1996:**

For every $\theta = a + b\sqrt{5}$, $a, b \in \mathbb{Q}$ there is an $s \in \mathbb{Z}$ such that $e^{2\pi i s\theta}$ is an eigenvalue of the Koopman operator of (I^*_{min}, F_π).

Are there rational eigenvalues?

For any $m \geq 1$, the components of $B^m_{min} h^{(1)}$ are odd, so $e^{2\pi i/2^k}$ is not an eigenvalue for any $k \geq 1$.

More generally, one can show there are no rational eigenvalues.

So (I^*_{min}, F_π) does not have the dyadic odometer as a factor, and it is also not weakly mixing.

Factors for (I^*_{np}, F_π)

We use the results of **Bezuglyi, Kwiatkowski, Medynets, Solomyak 2010** to determine eigenvalues for the non-minimal system (I^*_{np}, F_π).

Main differences with minimal case:

- There may be multiple ergodic measures with different eigenvalues.

- Eigenfunctions need not be automatically continuous.

Results: Similarly to the minimal case:

- There are rotated odometers which have the dyadic odometer as a maximal equicontinuous factor.

- There are systems where eigenvalues for different ergodic measures are different, and there are no continuous eigenvalues.

Example: Let $q = 5$ and $\pi = (01234)$, then $\{\chi_k\}_{k \geq 1}$ is constant.

Recall that there is an ergodic measure μ_2 supported on I_{np}^*, corresponding to the Lebesgue measure on I, and

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 4 & 8 & 8 & 4 & 8 \end{pmatrix}.$$

B has eigenvalues 8, 2 and 0 of multiplicity 3.

It follows that $e^{2\pi i/2^m}$ is an eigenvalue for any $m \geq 1$ with respect to μ_2.

Proposition: The dyadic odometer is the maximal equicontinuous factor for (I_{np}^*, F_π).

Example: Let $q = 5$ and let $\pi = (02431)$, then $\{\chi_k\}_{k\geq 1}$ is constant.

Recall that there is an ergodic measure μ_2 supported on I^*_{np}, corresponding to the Lebesgue measure on I, and

$$B = \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 5 & 3 & 8 & 5 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

B has eigenvalues 8, $2 \pm \sqrt{5}$, -1 of multiplicity 2.

Recall that the minimal subsystem has irrational eigenvalues $e^{2\pi s(a+b\sqrt{5})}$, $a, b \in \mathbb{Q}$, $s \in \mathbb{Z}$.

What about eigenvalues of (I^*_{np}, F_π, μ_2)?

Proposition: We have for (I^*_{np}, F_π, μ_2):

- $e^{2\pi i/2^m}$ is an eigenvalue if and only if $m = \{1, 2\}$.
- For any $a, b \in \mathbb{Q}$ the number $e^{2\pi i(a+b\sqrt{5})}$ is not an eigenvalue.
- For any other rational or irrational θ, $e^{2\pi i\theta}$ is not an eigenvalue.

Thus (I^*_{np}, F_π, μ_2) has the finite group with 4 elements as the only rotational factor, but the factor map is not continuous.

We have found examples of rotated odometers, where the minimal system (I^*_{min}, F_π) and the aperiodic subsystem (I^*_{np}, F_π) have the dyadic odometer as a factor, and examples, where they do not have the dyadic odometer as a factor.

But we can only determine that on a case-by-case basis.

Problem

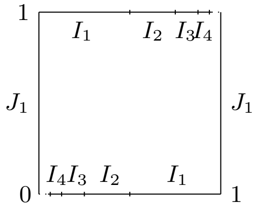
Let $F_\pi : I \to I$ be a rotated odometer. Find necessary and sufficient conditions under which (I^*_{min}, F_π) (or (I^*_{np}, F_π)) has a dyadic odometer as a factor.

Problem

Are there any rotated odometers for which Lebesgue or the measure on I_{min} are weakly mixing?

Loch-Ness monsters

Identify the horizontal edges minus discontinuity points by the von Neumann-Kakutani map, and vertical edges as in the standard torus.



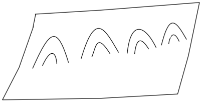
This construction is similar to that of the *Chamanara* or *baker's surface* in the literature.

In the Chamanara surface, the vertical sides are also identified using the von Neumann-Kakutani map.

We use arguments from **Randecker 2016 (thesis)** and **Delecroix, Hubert, Valdez (book)** to show that L has properties similar to the Chamanara surface, namely:

The resulting surface L is

- a translation surface (admits an atlas with translation transition maps),
- non-compact of finite area,
    - has infinite genus,
    - has a single end.



A **Loch-Ness monster** is a topological surface of infinite genus with a single end.

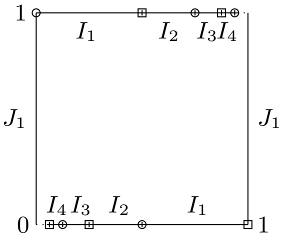
Loch-Ness monsters (usually of infinite area) also appear as typical leaves in foliations of compact manifolds by surfaces.

$\overline{L} \setminus L$ **contains a single point**

$\overline{L}$ is the metric completion of L.

Points in $\overline{L}$ marked by circles and squares are identified.

They are the same point since the distance between them is not bounded away from zero.

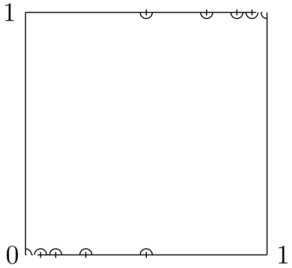


The point in $\overline{L} \setminus L$ is a *wild* singularity.

L has a single end

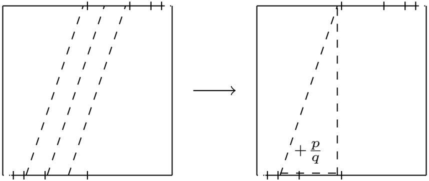
L has a single end which corresponds to a singularity.

Indeed, the complement of an open ϵ-neighborhood of the singularity is a compact subset of a square.



Straight-line flow on a Loch-Ness monster

Straight-line flow lines at angle $\theta = \tan^{-1}(q/p)$ travel through distance p/q in horizontally while traveling through distance 1 vertically.



Let P be a horizontal section minus points for which the flow is not defined for some $t \in \mathbb{R}$, $F : P \to P$ be first return map, λ be Lebesgue measure.

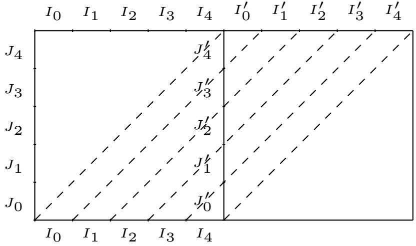
Lemma: (P, F, λ) is measurably isomorphic to the rotated odometer (I, F_π, λ) for π corresponding to a translation by p/q.

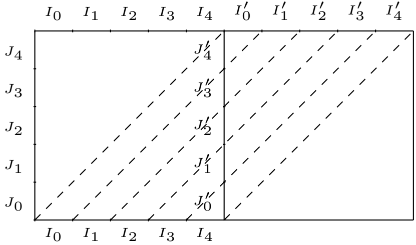
Let π be any permutation of q symbols, let $r \geq q$ and let

$$0 < \theta = \tan^{-1}\left(\frac{r}{q}\right) \leq \frac{\pi}{4},$$

so each flow lines intersects the square S in the horizontal direction completely at least once.

For simplicity, in the talk we consider the flow at angle $\frac{\pi}{4}$.





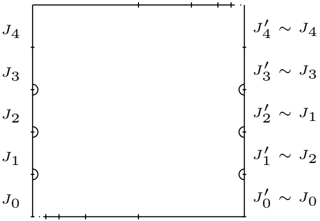
The flow induces the maps

$$\{I_j\} \xrightarrow{s} \{J'_j\} \xrightarrow{s^{-1}} \{I'_j\} \xrightarrow{\text{mod } 1} \{I_j\}$$

Set $\pi' = s \circ \pi \circ s^{-1}$ and identify by translations for $j = 0, \ldots, q-1$

$$J'_k \sim J_{\pi'(k)}.$$

Build a surface, where the horizontal sides are identified using the von Neumann-Kakutani map, and vertical sides using the permutation π'.



Example: If $\pi = (0)(1)(23)(4)$ and $\theta = \frac{\pi}{4}$ then $\pi' = (0)(12)(3)(4)$.

The surface has one wild singularity and one cone angle singularity of multiplicity 3.

It has one non-planar and one planar end.

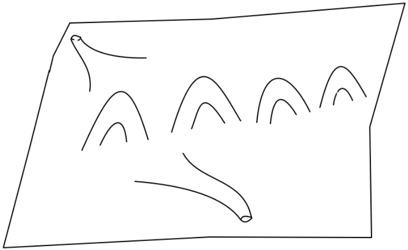
Theorem (Bruin and Lukina 2021)

Let $q \geq 2$, π be a permutation of q symbols, $r \geq q$ be an integer.
Let λ be Lebesgue measure on I.

There exists a translation surface $L_{\pi,r}$ such that

1. The surface $L_{\pi,r}$ has finite area, one non-planar end and at most a finite number of planar ends.
2. The metric completion of $L_{\pi,r}$ contains a single wild singularity and at most a finite number of cone angle singularities.
3. For a horizontal section $P \subset L_{\pi,r}$ with Poincaré map $F : P \to P$ of the flow of slope r/q, such that (P, F, λ) is measurably isomorphic to the rotated odometer (I, F_π, λ).

A Loch-Ness monster with a finite number of planar ends is a Loch-Ness monster with whiskers.



We showed that rotated odometers model flows of rational slope on some translation surfaces of infinite type.

The following problem is natural.

Problem

Find a Bratteli-Vershik system that models the first return (Poincaré) map of a flow of irrational slope on a translation surface of finite area with infinite genus and finite number of ends.

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Thank you for your attention!