Number of Ergodic and Generic Measures for Minimal Subshifts

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2021-01-12

Definition

- $\mathcal{A}$ is a finite *alphabet* of letters
- $\mathcal{A}^* = \{w = w_1 w_2 \dots w_n : \ w_i \in \mathcal{A}, \ n \in \mathbb{N}\}$ finite words
- A *language* $\mathcal{L} \subseteq \mathcal{A}^*$ is any set such that

$$\forall w \in \mathcal{L}, \ \exists a, b \in \mathcal{A}, \ \text{s.t.} \ awb \in \mathcal{L}$$

and $\forall w = w_1 \dots w_n \in \mathcal{L}$, $1 \le j \le k \le n$,

$$w_{[j,k]} = w_j w_{j+1} \dots w_{k-1} w_k \in \mathcal{L}.$$

Definition

The *(factor/subword) complexity* function for $\mathcal{L}$ is

$$p(n) = \Big|\underbrace{\{w \in \mathcal{L} : \ |w| = n\}}_{\mathcal{L}_n}\Big|,$$

where $|w|$ is the length of w.

Definition

The subshift (X, T) with language $\mathcal{L}$ is the set

$$X = \{x = x_1 x_2 \cdots \in \mathcal{A}^{\mathbb{N}} : \; x_{[1,n]} \in \mathcal{L} \quad \forall n \in \mathbb{N}\}$$

the left shift map $T : X \to X$,

$$T(x_1 x_2 x_3 \dots) = x_2 x_3 x_4 \dots.$$

X is endowed with the topology (and Borel σ-algebra) generated by the cylinder sets

$$[w] = \{x \in X : \; x_{[1,n]} = w\}$$

for each $w = w_1 w_2 \dots w_n \in \mathcal{L}$.

The set $\mathcal{M}(X)$ of shift invariant probability measures:

- Has a convex structure, meaning

$$\mu, \nu \in \mathcal{M}(X), t \in [0,1] \Rightarrow t \cdot \mu + (1-t) \cdot \nu \in \mathcal{M}(X).$$

- The ergodic measures $\mathcal{E}(X) \subset \mathcal{M}(X)$ are the *extremal* elements, meaning

$$\rho = t\mu_1 + (1-t)\mu_2 \Rightarrow \mu_1 = \mu_2 = \rho$$

  for $\rho \in \mathcal{E}(X)$, $\mu_1, \mu_2 \in \mathcal{M}(X)$ and $t \in (0,1)$.

- By the Pointwise Ergodic Theorem, if $\rho \in \mathcal{E}(X)$ then for ρ-a.e. $x \in X$ we have

$$\frac{\left|x_{[1,N]}\right|_w}{N} \xrightarrow[N\to\infty]{} \rho\big([w]\big) \quad \forall w \in \mathcal{L}.$$

  We call such an x a ρ-generic point.

- The set of generic measures $\mathcal{G}(X) \subset \mathcal{M}(X)$ are the elements $\rho \in \mathcal{M}(X)$ such that a ρ-generic point $x \in X$ exists.
- So

$$\mathcal{E}(X) \subset \mathcal{G}(X) \subset \mathcal{M}(X).$$

  None necessarily equal.

Question

What relationship(s) exist between $\mathcal{L}$ and the sets $\mathcal{E}(X), \mathcal{G}(X)$ of measures?

Theorem (F. in preparation)

Let $\mathcal{L}$ be uniformly recurrent and

$$\lim_{n\to\infty} \frac{p(n)}{n} = K,$$

for some $K \geq 4$. Then $|\mathcal{G}(X)| \leq K - 2$.

- If this limit exists, then $K \in \mathbb{N}$ (Cassaigne & Nicolas 2010).
- Uniform recurrence: $\forall n,\ \exists N$ s.t. each $W \in \mathcal{L}_N$ contains each $w \in \mathcal{L}_n$ as a subword. *Equivalent to minimality of (X, T).*
- The $\limsup$ and $\liminf$ of $\frac{p(n)}{n}$ are invariant under topological conjugacy.

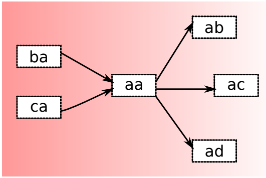
Definition

A word $w \in \mathcal{L}$ is:

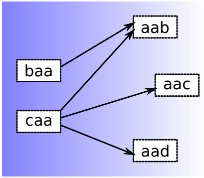
- *left (resp. right) special* if there exist more than one left (resp. right) extension of w in $\mathcal{L}$.
- *bispecial* if it is left and right special
- *regular bispecial* if it is bispecial and exactly one left (resp. right) extension is right (resp. left) special.

Example with Rauzy Graphs

(part of) Γ_2



(part of) Γ_3



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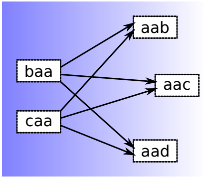


In the language creating the top Γ_3, *aa* is regular bispecial.
In the language creating the bottom Γ_3, *aa* is bispecial but not regular bispecial.

Definition

$\mathcal{L}$ satisfies the *Regular bispecial condition (RBC)* if all sufficiently long bispecial words are regular.

Lemma

If $\mathcal{L}$ satisfies the RBC, there exists a growth rate K such that

$$p(n) = Kn + C \quad \forall n \geq n_0 \tag{ECG}$$

for integers C, n_0.

Theorem (Damron & F. accepted)

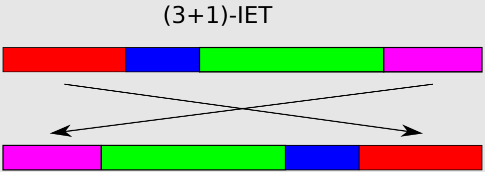
Let $\mathcal{L}$ be recurrent satisfying the RBC with growth rate K. Then

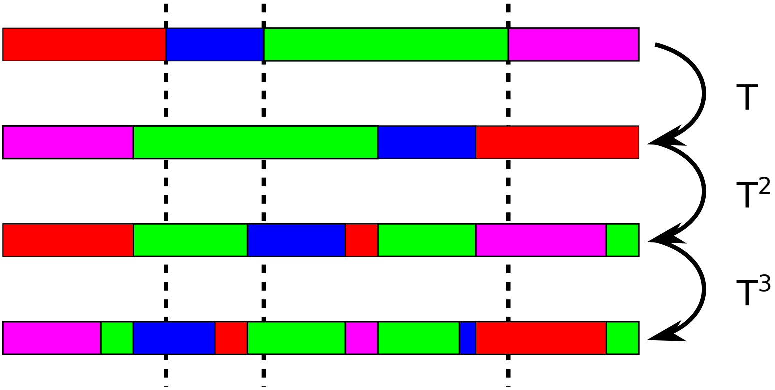
$$\left|\mathcal{E}(X)\right| \leq \frac{K+1}{2}.$$

- Recurrence: $\forall u, v \in \mathcal{L}$, $\exists w \in \mathcal{L}$ such that $uwv \in \mathcal{L}$. *Equivalent to topological transitivity of* (X, T).
- The RBC is equivalent to *eventual dendricity* (defined by Dolce & Perrin 2019), so
    - The RBC is invariant under topological conjugacy.
    - Recurrence is equivalent to uniform recurrence.

"Definition"

An *Interval Exchange Transformation* on $(K+1)$ intervals is defined by an ordering π on $\{1, 2, \ldots, K+1\}$ and a choice $\lambda = (\lambda_1, \ldots, \lambda_{K+1})$ of sub-interval lengths.





A typical IET is minimal (so not periodic).

Theorem (Katok 1973, Veech 1978)

If $(K+1)$-IET is minimal, then $|\mathcal{E}(X)| \leq \frac{K+1}{2}$.*
* *The bound depends on π that defines the IET and may be strictly less than $\lfloor \frac{K+1}{2} \rfloor$.*

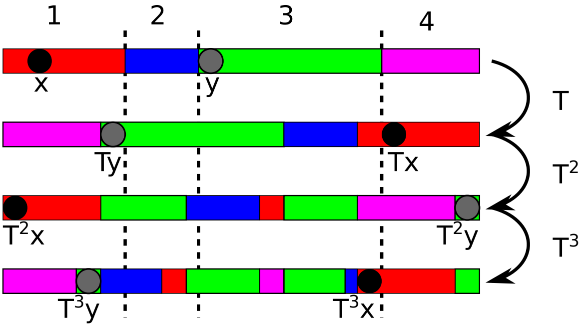
Example (Keane 1977, Yoccoz 2009, F. 2014)

This bound is sharp, i.e. there exist minimal IETs that achieve the known bound for each choice of irreducible π.

Both proofs use the geometry of associated surfaces in some form.

"Natural" Coding to IET subshift

$w = w_1 \dots w_n \in \mathcal{L}$ iff there exists point x on the interval such that for all $0 \leq k \leq n-1$ we have $T^k x$ in sub-interval labeled w_k.



In this example, $1413 \in \mathcal{L}$ and $3141 \in \mathcal{L}$

As a subshift, the typical $(K+1)$-IET has complexity

$$p(n) = Kn + 1.$$

Ferenczi & Zamboni (2008) characterized all $\mathcal{L}$ that arise naturally from IETs.

Question (Boshernitzan 1984/85)

Can the bound $|\mathcal{E}(X)| \leq \frac{K+1}{2}$ be shown using $\mathcal{L}$?

Theorem (Boshernitzan 1984/85)

Let $\mathcal{L}$ be uniformly recurrent.

- *If* $\liminf_{n\to\infty} \frac{p(n)}{n} = \alpha$, *then* $|\mathcal{E}(X)| \leq \max\{1, \lfloor \alpha \rfloor\}$
- *If* $\limsup_{n\to\infty} \frac{p(n)}{n} = \alpha$, *then* $|\mathcal{E}(X)| \leq \max\{1, \lfloor \alpha \rfloor - 1\}$

For a $(K+1)$-IET, this implies $|\mathcal{E}(X)| \leq K - 1$, which agree with the known bound for $K \leq 3$ (or up to 4 intervals).

Question

Can these bounds be improved for such subshifts?

Answer (No)

(Cyr & Kra 2019) For each $K \in \mathbb{N}$, $K \geq 2$, there exists uniformly recurrent $\mathcal{L}$ such that

$$\liminf_{n\to\infty} \frac{p(n)}{n} = K - 1, \ \limsup_{n\to\infty} \frac{p(n)}{n} = K \text{ and } |\mathcal{E}(X)| = K - 1.$$

We need to restrict further!

Earlier Progress

Theorem (Damron & F. 2017)

Let $\mathcal{L}$ be uniformly recurrent such that for $K, C, n_0 \in \mathbb{N}$, $K \geq 4$,

$$p(n) = Kn + C, \quad \forall n \geq n_0, \tag{ECG}$$

then $|\mathcal{E}(X)| \leq K - 2$.

When combined with previous results, the bound for $(K+1)$-IETs is agrees with known bound for $K \leq 5$ (up to six intervals).

Question

Is this bound sharp?

The properties for $\mathcal{L}$ from a $(K+1)$-IET by (Ferenczi & Zamboni 2008) imply the RBC that we use in our proof that $\left|\mathcal{E}(X)\right| \leq \frac{K+1}{2}$.

What about $\mathcal{G}(X)$?

Example (Chaika & Masur 2015)

There exists a $(5+1)$-IET such that $|\mathcal{E}(X)| = 2$ and $|\mathcal{G}(X)| = 3$.

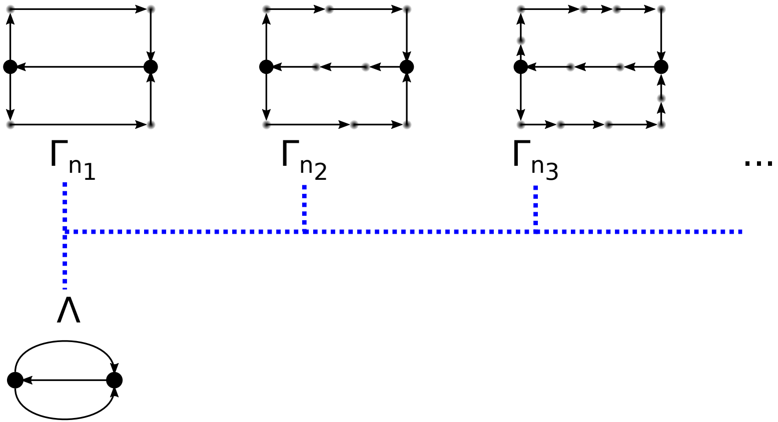
Question (Chaika & Masur 2015)

Is it true that $|\mathcal{G}(X)| \leq \frac{K+1}{2}$ *for minimal* $(K+1)$*-IETs?*

Cyr & Kra (2019) proved Boshernitzan's bounds, but for $\mathcal{G}(X)$ and with relaxed conditions on $\mathcal{L}$ (uniform recurrence not required).

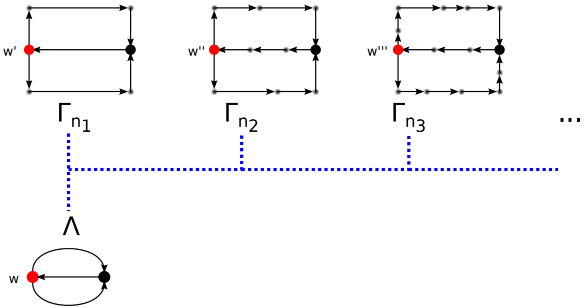
Main Idea

Draw and color!



We create a (multi)graph Λ based on the branching points (special words) along the sequence $\{\Gamma_n\}$ Rauzy Graphs.

We create a (non-standard) coloring rule $\mathcal{C}$ on Λ.
STEP 1: Assign to $w \in \Lambda$ a measure $\mu_w \in \mathcal{M}(X)$.



$$\left[\quad \forall v \in \mathcal{L}, \quad \mu_w([v]) = \lim_{k\to\infty} \frac{|w^{(k)}|_v}{|w^{(k)}|} \quad \right]$$

STEP 2: For each $\rho \in \mathcal{E}(X)$ fix ρ-generic point $x^{(\rho)} \in X$.

STEP 3: Check a density function $\mathcal{D}$ defined by the frequency of the $w^{(k)}$'s within $x^{(\rho)}$.

STEP 4: If $\mathcal{D} > 0$ then $\rho \geq \delta \cdot \mu_w$ for $\delta = \delta(\mathcal{D}, K) \in (0, 1)$.

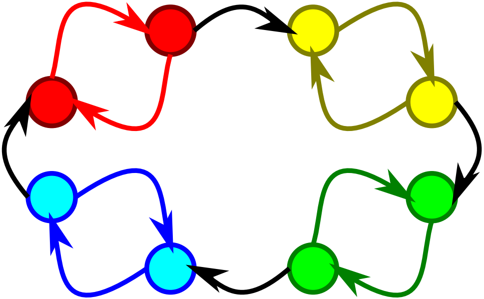
STEP 5: Extremality of $\mathcal{E}(X)$ implies that $\rho = \mu_w$, so we assign w the "color" ρ.

Lemma

- *$\forall \rho \in \mathcal{E}(X)$, there exists vertices/edges colored with ρ by $\mathcal{C}$.*
- *Edge and vertex colorings are consistent.*
- *If an egde is colored by ρ, it belongs to a ρ-colored circuit.*

There may be edges/vertices that are not colored.

We need more!

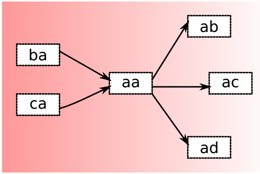


$$\Lambda$$

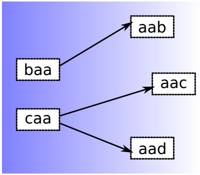
Here, there are $K = 4$ colors.

Bispecial moves control evolution of Γ_n's as n increases.

Γ_n

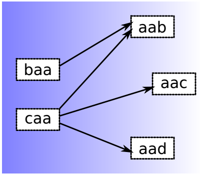


Γ_{n+1}



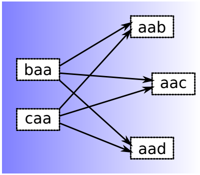
(not RBC)

or



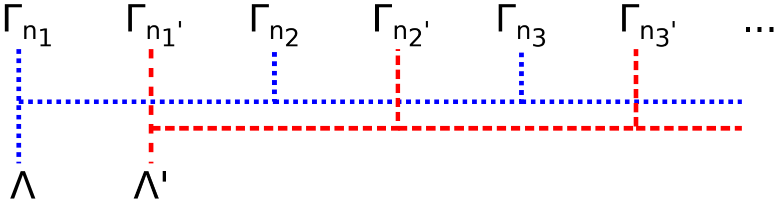
(RBC)

or

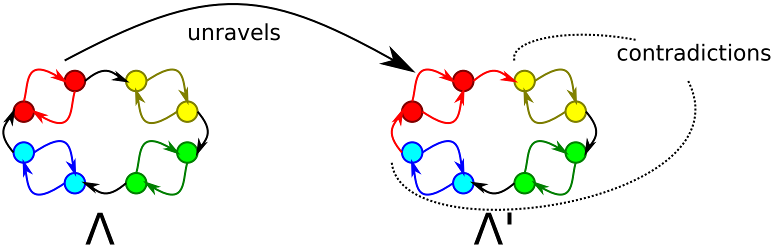


(not RBC)

These moves define evolutions from Λ to new Λ' with consistent colorings $\mathcal{C}$ and $\mathcal{C}'$ (resp.).

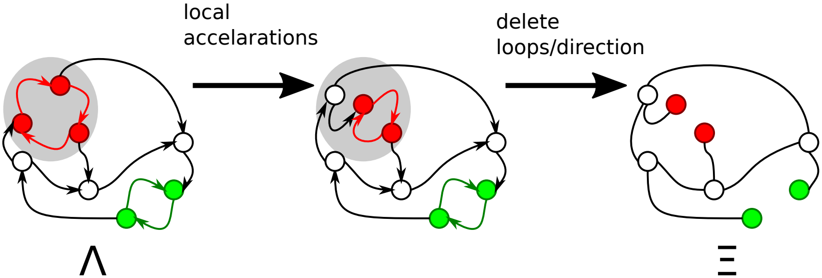


Loops must “unravel” and therefore spread color.



We actually get $|\mathcal{E}(X)| \leq K - 2$ for $K \geq 4$ here (Damron & F. 2017). *Need more to significantly improve!*

We accelerate fixed colored loops asynchronously and then remove them to create a new (undirected) graph Ξ.



Lemma (Main Lemma)

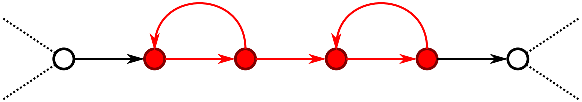
If $\mathcal{L}$ satisfies the RBC, Ξ must be weakly connected.

Consequence

$$|\Xi|_{\text{edges}} \geq |\Xi|_{\text{vertices}} - 1$$

- Previous coloring method does not work for $\mathcal{G}(X)$, as the elements of $\mathcal{G}(X) \setminus \mathcal{E}(X)$ are not extremal.
- For fixed proportion $D \gg K$, for all large enough n there exists a walk $W_{\rho,n} \subset \mathcal{L}_n$ in Γ_n of length at least Dn.
- These walks $W_{\rho,n}$ and $W_{\rho',n}$, $\rho \neq \rho'$, are vertex disjoint.

- We may then "color" vertices and edges along these walks.
- These colorings must still contain loops, but do not enjoy all properties of the coloring rule from $\mathcal{E}(X)$.



- Not as "acceleration friendly" but still retains relationships between $W_{\rho,n}$ and $W_{\rho,n'}$ for $n' > n$.

Details

STEP 1: For $\rho, \rho' \in \mathcal{G}(X)$, $\rho \neq \rho'$, $\exists\ w = w_{\{\rho,\rho'\}}$ such that

$$\rho([w]) \neq \rho'([w])$$

STEP 2: There exists n_0 such that for all $N \geq n_0$

$$\left| \frac{\left|x^{(\rho)}_{[1,N]}\right|_w}{N} - \rho([w]) \right| \ll |\rho([w]) - \rho'([w])|$$

where $\ll$ depends on the fixed $D \gg K$(similar for ρ' with same n_0)

Details

STEP 3: For each n and $n_0 \leq m \leq Dn$,

$$\left| \frac{|x^{(\rho)}_{[m,m+n-1]}|_w}{n} - \rho([w]) \right| < \frac{|\rho([w]) - \rho'([w])|}{3}$$

(similar for ρ')

STEP 4: So

$$W_{\rho,n} = \{x^{(\rho)}_{[m,m+n-1]} : n_0 \leq m \leq Dn\}$$

and $W_{\rho,n} \cap W_{\rho',n} = \emptyset$.

Examples?

- For $K \geq 4$, do there exist uniformly recurrent $\mathcal{L}$ such that $p(n) = Kn + C$ for all large n such that $|\mathcal{E}(X)| = K - 2$? If not, what if $\frac{p(n)}{n} \to K$ is assumed instead?
- Do there exist examples with this constant growth condition that are uniformly recurrent, satisfying $|\mathcal{E}(X)| \geq 2$ but not from an IET?

(More) Examples?

- Do there exist uniformly recurrent $\mathcal{L}(X)$ such that $\liminf_{n\to\infty} \frac{p(n)}{n} = K - 1$ and $\limsup_{n\to\infty} \frac{p(n)}{n} = K$ and $|\mathcal{G}(X)| = K - 1$ but $|\mathcal{G}(X) \setminus \mathcal{E}(X)| = G$, for fixed $1 \leq G \leq K - 3$?
- Similar to above, but assuming $p(n) = Kn + C$ (eventually) or $\lim_{n\to\infty} \frac{p(n)}{n} = K$ for $K \geq 4$.

Further Proofs?

- Can we use the remaining Ferenczi & Zamboni (2008) properties to fully achieve the known bound* on $|\mathcal{E}(X)|$ for minimal IETs?
- Can we use word combinatorics to provide a proof on the bound of $|\mathcal{G}(X)|$ for minimal IETs?

*For example if $\pi = (9, 4, 3, 2, 5, 8, 7, 6, 1)$ the known bound for $|\mathcal{E}(X)|$ is $3 < \left\lfloor \frac{9}{2} \right\rfloor$.

Merci pour votre attention!

Thank you!