Recent developments in finite rank systems

Sebastián Donoso

Center for Mathematical Modeling and Department of Mathematical Engineering
University of Chile

Invariants combinatoire et algébriques des décalages et des pavages
11 January 2021

Basic definitions

Let $\mathcal{A}$ be a finite alphabet.

Let $S\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, $(x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}$ be the *shift*.

A closed subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$, invariant under S is a subshift. We write (X, S) to denote a subshift.

A word $w \in \mathcal{A}^n$ appears in X if there exists $x \in X$ and $i \in \mathbb{Z}$ such that $\underline{w = x_{[i,i+n-1]}}$

$\mathcal{L}_n(X) = \{w \in \mathcal{A}^n : w \text{ appears in } X\}$.

$p_X(n) = |\mathcal{L}_n|$. complexity function

$$h_{top}(X,S) = \lim_{n\to\infty} \frac{\log(p_X(n))}{n}$$

Some motivations

Systems with non-superlinear complexity exhibit rigidity properties.

$$\liminf \frac{P_X(n)}{n} < +\infty$$

1) Zero entropy

2) Finite number of ergodic measures
   - Boshernitzan (1984)
   - Cyr-Kra (2019)
   - Damron, Fickenscher (2017) (Talk tomorrow)
   - Dykstra, Ormes, Pavlov (2019)

3) Constrained automorphism group $Aut(X,S) = \{ h \text{ homeo}, h\circ S = S \circ h \}$

- Salo - Törmä (2013)
- Cyr - Kra
- D. Durand, Maass, Petite
- Coven, Quas, Yassawi

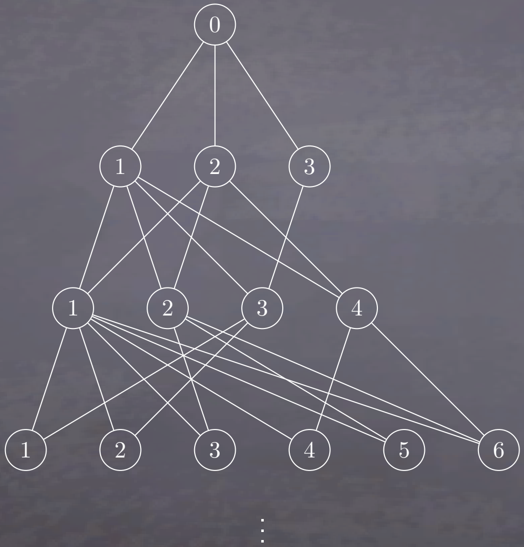
$\Rightarrow \quad \mathrm{Aut}(X,S) / \langle S \rangle$ finite

$\boxed{\liminf \dfrac{P_X(n)}{n} < +\infty}$

3') Finite number of asymptotic components

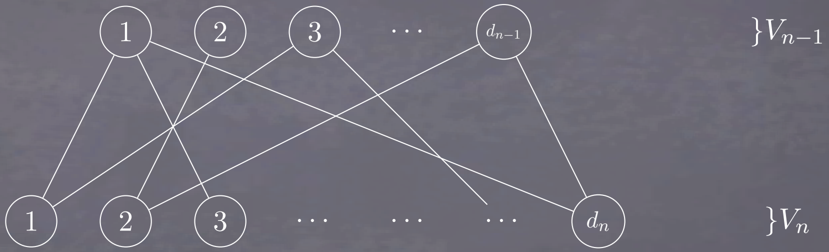
More definitions: ~~Bratelli~~-Vershik diagrams

Brattelli

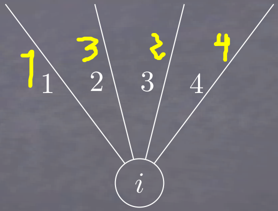


$$1 \quad 2 \quad 3 \quad \cdots \quad d_{n-1} \qquad \}V_{n-1}$$

$$1 \quad 2 \quad 3 \quad \cdots \quad \cdots \quad \cdots \quad d_n \qquad \}V_n$$

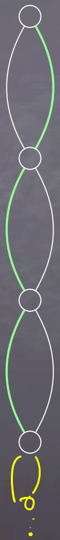

1
2
3
⋯
d_{n-1}
$\}V_{n-1}$
1
2
3
⋯
⋯
⋯
d_n
$\}V_n$

There is an order on the edges arriving to a vertex, for every vertex.



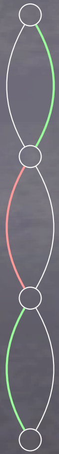
Bratelli-Vershik map

Let X be the set of all infinite paths. We can define $B_V : X \to X$ the successor map.



Bratelli-Vershik map

Let X be the set of all infinite paths. We can define $B_V : X \to X$ the successor map.

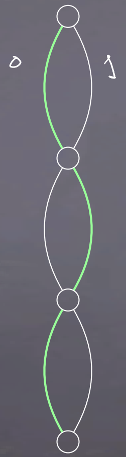


Bratelli-Vershik map

Let X be the set of all infinite paths. We can define $B_V : X \to X$ the successor map.

Bratelli-Vershik map

Let X be the set of all infinite paths. We can define $B_V : X \to X$ the successor map.

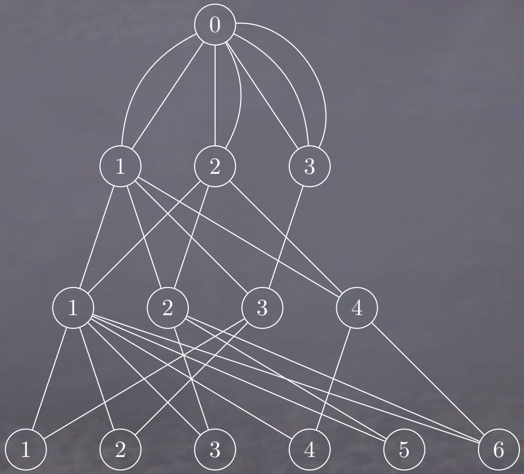


$$\left.\begin{array}{l} 100\ldots\ldots \\ 010\ldots\ldots \end{array}\right\} +1$$

Odometer

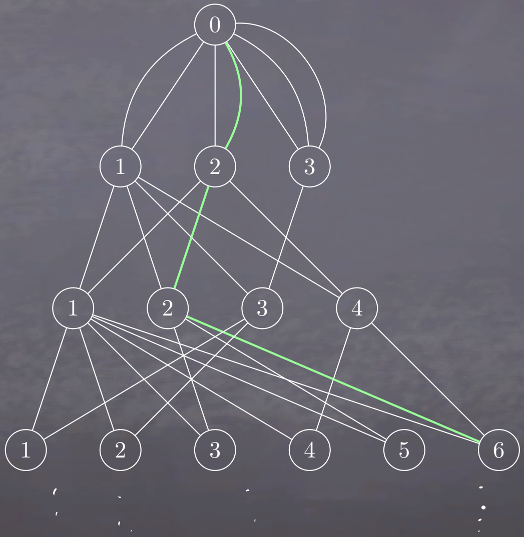
Bratelli-Vershik map

Let X be the set of all infinite paths. We can define $B_V : X \to X$ the successor map.



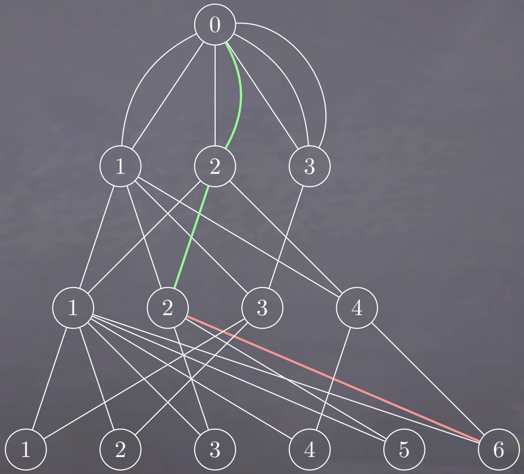
Bratelli-Vershik map

Let X be the set of all infinite paths. We can define $B_V : X \to X$ the successor map.



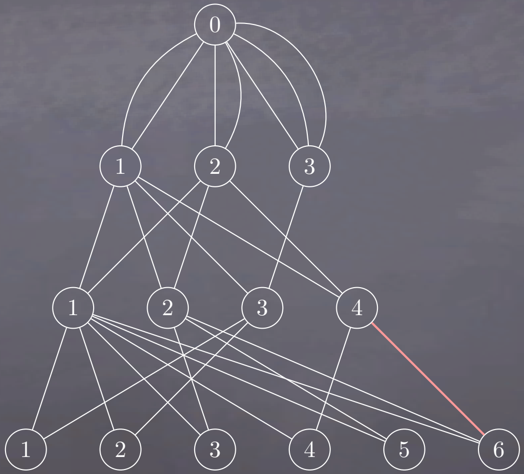
Bratelli-Vershik map

Let X be the set of all infinite paths. We can define $B_V : X \to X$ the successor map.



Bratelli-Vershik map

Let X be the set of all infinite paths. We can define $B_V : X \to X$ the successor map.



Bratelli-Vershik map

Let X be the set of all infinite paths. We can define $B_V : X \to X$ the successor map.

More definitions/theorems

The diagram is simple if for each node of level n is connected to all nodes from some level $N \geq n$.

Such dynamics are minimal (all orbits are dense).

Theorem (Herman, Putnam and Skau, 1992)

cantor space

Any minimal Cantor system (X, T) is topologically conjugate to a Bratteli-Vershik system (X_B, V_B) where (V, E) is simple.

Definition

A system is of finite rank if it can be represented with a B-V diagram with $|V_n|$ bounded.

Theorem (Downarowicz and Maass, 2008.)

Every minimal finite rank Bratelli Vershik diagram is either expansive (a subshift) or equicontinuous (an odometer).

Rigidity properties of minimal finite rank systems

Theorem (Bezuglyi, Kwiatkowski, Medynets, Solomyak, 2012.)

Ergodic measures : finite n°. (less than rank)

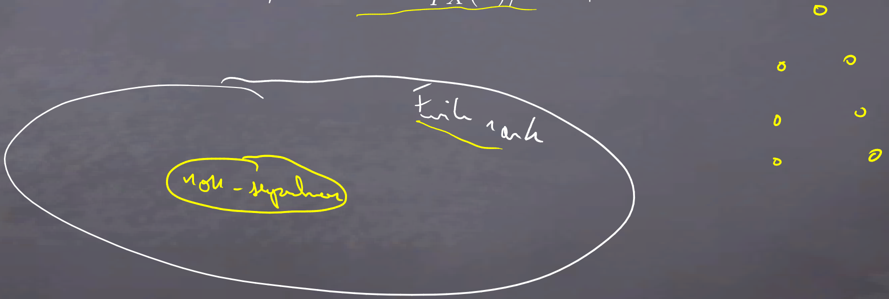
Entropy : 0 entropy

Some results

<u>Theorem</u> (D., Durand, Maass, Petite, 2020)

Let (X, S) be a minimal subshift. If $\liminf p_X(n)/n < +\infty$ then (X, S) is of finite rank.

There are minimal rank 2 subshift, with $\liminf p_X(n)/n = +\infty$.



Some (already solved) questions

Question

Does a minimal finite rank system have finitely many asymptotic components? $\Rightarrow$ $\frac{\mathrm{Aut}(X,S)}{\langle S \rangle}$ finite

B. Espinoza (2020)

Question

Is a (subshift) factor of a finite rank subshift a finite rank subshift?

Espinoza (2020)
Golestani, Hosseini

Question

Does a minimal finite rank system have a finite number of (symbolic) factors?

Espinoza

Talks on Thursday

Complexity of F-R systems?

$\mathcal{S}$-adic subshifts

Morphisms

$\tau : A^* \to B^*$, $\tau(a_1 a_2 a_3 \cdots a_n) = \tau(a_1)\tau(a_2)\cdots\tau(a_n)$

$\tau(a) = b_1 b_2 \cdots b$

Let $\tau_k : \mathcal{A}_{k+1}^* \to \mathcal{A}_k^*$ positive morphisms. Denote $\boldsymbol{\tau} = (\tau_k)_{k\in\mathbb{N}}$ and $\tau_{[a,b]} = \tau_a \circ \tau_{a+1} \cdots \tau_b$.

$X_{\boldsymbol{\tau}}$ is the associated $\mathcal{S}$-adic subshift. A point $x = (x_i)_{i\in\mathbb{Z}} \in X_{\boldsymbol{\tau}}$ iff

any subword of x is a subword of $\tau_{[0,N]}(a)$ for some $N \in \mathbb{N}$ and $a \in \mathcal{A}_{N+1}$.

$\cdots \quad \tau_k : A_{k+1} \longrightarrow A_k \longrightarrow \cdots A_1 \longrightarrow A_0 \qquad x \in A_0^{\mathbb{Z}}$

$a \to \tau_k \circ \cdots \tau_0(a)$

$x \quad \cdots$

Alphabet rank

$$AR(\boldsymbol{\tau}) = \liminf_{k \to +\infty} |\mathcal{A}_k|.$$

Expansive finite rank systems are of this type.

Complexity of $\mathcal{S}$-adic subshifts.

Question

How to estimate the complexity $p_{X_\tau}(n)$*?*

Some known cases

Substitutions $\tau = \tau_0$

sublinear $p(n) \leq Cn$

$\cdot$

$\cdot$

Linearly recurrent.

Finite set of morphisms

sublinear $P(n) \leq Cn$

Finite rank.

subexponential

A notion of complexity of a morphism

For a morphism $\tau\colon \mathcal{A}^* \to \mathcal{B}^*$ set

$$\|\tau\| = \max_{a\in\mathcal{A}} |\tau(a)| \text{ and } \langle\tau\rangle = \min_{a\in\mathcal{A}} |\tau(a)|.$$

a
b
c

For $a \in \mathcal{A}$ write

$$\tau(a) = b_{i_1}^{l_1} b_{i_2}^{l_2} \cdots b_{i_{k(a)}}^{l_{k(a)}}, \quad b_{i_k} \neq b_{i_{k+1}}$$

and define

$$\boxed{\text{r-comp}(\tau) = \sum_{a\in\mathcal{A}} k(a).}$$

the repetition complexity.

$a \longrightarrow a\,b\,c\,a\,bbbbbba$ } 6

1 +1 +1 +1 +1 +1

by

{ 7

Proposition

Let (X, S) be the $\mathcal{S}$-adic subshift generated by the positive directive sequence $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n \geq 0}$. Then, we have that

$$p_X(n) \leq \underbrace{(\max_{i \in \mathbb{N}} |\mathcal{A}_i| + 2)}\underbrace{(\limsup_{i \to +\infty} \text{r-comp}(\tau_i) + 1)} \cdot n$$

for all large enough $n \in \mathbb{N}$.

Proposition

Let (X, S) be a finite topological rank minimal Cantor system. Then (X, S) is strongly orbit equivalent to a subshift of sublinear complexity.

Relative complexity of morphisms

Definition

Let $\sigma\colon \mathcal{B}^* \to \mathcal{C}^*$ be a morphism. For $b \in \mathcal{B}$ and $\mathcal{L} \subseteq B^*$, define the following subset of words that extends b to the right.

$$\mathcal{F}(b, n, \sigma, \mathcal{L}) = \left\{ w \in \mathcal{L} : bw \in \mathcal{L}, w_1 \neq b, |\sigma(w_{[1,|w|-1]})| < n \leq |\sigma(w)| \right\}.$$

That is, each word in $\mathcal{F}(b, n, \sigma, \mathcal{L})$ extends to the right some appearance of b in $\mathcal{L}$, starts with a letter different from b and has the shortest image under σ of length at least n.

Definition

Let $\tau\colon \mathcal{A}^* \to \mathcal{B}^*$ and $\sigma\colon \mathcal{B}^* \to \mathcal{C}^*$ be two morphisms. For $i \in \mathbb{N}^*$, we define the *i-th complexity map of σ with respect to τ* as

$$P^{(i)}_{\sigma|\tau}(n) = \sum_{b \in \mathcal{B}} |\mathcal{F}(b, n, \sigma, \tau(\mathcal{A}^i))|.$$

The *complexity map of σ with respect to τ* is defined by

$$P_{\sigma|\tau}(n) = \sum_{b \in \mathcal{B}} |\mathcal{F}(b, n, \sigma, \mathcal{L}(\tau(\mathcal{A}^{\mathbb{Z}})))|.$$

Proposition

Let $\tau\colon \mathcal{A}^* \to \mathcal{B}^*$ *and* $\sigma\colon \mathcal{B}^* \to \mathcal{C}^*$ *be two morphisms and suppose that* τ *is positive. Then we have that*

$$p_{\sigma\circ\tau(\mathcal{A}^{\mathbb{Z}})}(n) \leq (|\mathcal{B}| + P^{(2)}_{\sigma|\tau}(n)) \cdot n$$

for all $n \in [\|\sigma\|, \langle \sigma \circ \tau \rangle]$.

Applications

Corollary

Let (X,S) be a $\mathcal{S}$-adic subshift generated by the positive directed sequence $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$. If $\liminf_{n\to+\infty} \frac{\log(|\mathcal{A}_{n+1}|)}{\langle \tau_{[0,n]} \rangle} = 0$, then (X,S) has zero entropy, i.e., $\lim_{n\to 1} \log(p_X(n))/n = 0$.

In particular, if (X,S) is a minimal Cantor system of finite topological rank, then it has zero entropy.

Proposition

If $\liminf_{k\to+\infty} |\mathcal{A}_k| \leq 2$*, then* (X, S) *is sub-quadratic along a subsequence, i.e.,*

$$\liminf_{n\to+\infty} \frac{p_X(n)}{n^2} = 0.$$

Questions and possible applications

Question

For a finite rank $\mathcal{S}$-adic, does there exists $d = d(rank)$ such that

$$\liminf_{n\to\infty} p_X(n)/n^d = 0.$$

Question

Let (X, S) be a Toeplitz subshift. Is it true that X has finite topological rank if and only if the complexity of X is non-superlinear?

Question

Let (X, S) be a finite rank $\mathcal{S}$-adic subshift. Can (X, S) be mixing for an invariant measure $\mu \in \mathcal{M}(X.S)_,$.

Question

When an $\mathcal{S}$-adic subshift has countably many ergodic measures? Can this be provided in "combinatorial terms"?

Question

Construct $\mathcal{S}$-adic subshifts with low superlinear complexity, having:

- *any given simplex of invariant measures.*
- *an automorphism group containing all finite groups.*
- *mixing for an invariant measure.*

Thanks!