# Weakly linear systems for matrices over the max-plus quantale and their applications


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# Weakly linear systems – origins

## Origins

- ⋆ ***Weakly linear systems*** (abbr. **WLS's**) – particular systems of matrix inequations
- ⋆ emerged in Fuzzy Set Theory – in the study of ***fuzzy finite automata*** (abbr. **FFAs**)

## Use

- ⋆ **state reduction** for FFAs
  - Ćirić, Stamenković, Ignjatović, Petković, J COMPUT SYST SCI 76 (2010) 609–633
  - Stamenković, Ćirić, Ignjatović, INF SCI 275 (2014) 168–198
- ⋆ **simulations and bisimulations** for FFAs
  - Ćirić, Ignjatović, Damljanović, Bašić, FUZZY SETS SYST 186 (2012) 100–139
  - Ćirić, Ignjatović, Jančić, Damljanović, FUZZY SETS SYST 208 (2012) 22–42
- ⋆ **simulations and bisimulations** for NFAs
  - Ćirić, Ignjatović, Bašić, Jančić, INF SCI 261 (2014) 185–218

**complete residuated lattices** – $\mathbb{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$

- ⋆ $(L, \vee, \wedge, 0, 1)$ – complete lattice, $\mathbf{0}$ – the least element, $\mathbf{1}$ – the greatest element
- ⋆ $(L, \otimes, 1)$ – commutative monoid
- ⋆ *residuation property* – $x \otimes y \leqslant z \quad \Leftrightarrow \quad x \leqslant y \rightarrow z$

**fuzzy matrices** – matrices with entries in $\mathbb{L}$

- ⋆ order on $\mathbb{L}$ pointwise extended to fuzzy matrices – to each $\mathbb{L}^{m\times n}$
- ⋆ $\vee$-$\otimes$-matrix multiplication – product of matrices over the semiring $(L, \vee, \otimes, 0, 1)$

**WLS's for fuzzy matrices**

$$
\begin{aligned}
A_i X &\leqslant X B_i\,, & i &\in I_1\\
X B_i &\leqslant A_i X\,, & i &\in I_2\\
X^\top A_i &\leqslant B_i X^\top\,, & i &\in I_3\\
B_i X^\top &\leqslant X^\top A_i\,, & i &\in I_4
\end{aligned}
$$

$\{A_i\}_{i\in I} \subset \mathbb{L}^{m\times m}$, $\{B_i\}_{i\in I} \subset \mathbb{L}^{n\times n}$, $I = I_1 \cup I_2 \cup I_3 \cup I_4$, $X$ – unknown matrix taking values in $\mathbb{L}^{m\times n}$

# Weakly linear systems – homogeneous and heterogeneous

- ***homogeneous WLS*** - $m = n$ (square matrices) and $A_i = B_i$ for each $i \in I$

  used in the **state reduction** of FFAs

  general properties of homogeneous WLS's

  Ignjatović, Ćirić, Bogdanović, FUZZY SETS SYST 161 (2010) 3081–3113

- ***heterogeneous WLS*** – rectangular matrices

  used in the study of **simulations** and **bisimulations** for FFAs

  general properties of heterogeneous WLS's

  Ignjatović, Ćirić, Damljanović, Jančić, FUZZY SETS SYST 199 (2012) 64–91

- other applications
  - *Social Network Analysis* – **positional analysis** of fuzzy social networks (FSNs), simulations and bisimulations for FSNs
  - *Multivalued Modal Logics* – simulations and bisimulations for **multivalued Kripke models**

- ⋆ **residuation** in $\mathbb{L}$ ➠ transferred to matrices
  $A\backslash C = \bigvee\{X \in \mathbb{L}^{m\times n} \mid AX \leqslant C\},\ C/B = \bigvee\{Y \in \mathbb{L}^{m\times n} \mid YB \leqslant C\},\ A \in L^{m\times m},\ B \in L^{n\times n},\ C \in L^{m\times n}$
  ➠ reduces the WLS to an inequation of the form $X \leqslant \phi(X)$, where $\phi$ is a particular isotone function on $\mathbb{L}^{m\times n}$

- ⋆ **completeness** of $\mathbb{L}$ ➠ transferred to matrices
  ➠ creates conditions for using the ***Knaster-Tarski fixed-point theorem***
  ➠ provides the ***existence of the greatest solution*** to the WLS

- ⋆ ***computing the greatest solution***
  ➠ by means of a modification of the ***Kleene fixed-point theorem***

- ⋆ **application to FFAs** ☜ ***compatibility*** of the ***matrix ordering*** with ***matrix multiplication***
- ⋆ ***compatibility*** ☜ ***positivity*** of the ordering in $\mathbb{L}$

# Weakly linear systems – for matrices over a semiring

- ★ *How to extend WLS's to the context of matrices over a semiring?*
- ★ *How to apply them to general* ***WFAs over a semiring*** *?*

- ★ ***Residuation?***
- ★ *additively idempotent semirings* ☞ ***relative residuation*** ☞ ***Boolean residuals*** *– Boolean matrices*

- ★ *Boolean simulations and bisimulations for* ***WFAs over additively idempotent semirings***

Damljanović, Ćirić, Ignjatović, THEOR COMPUT SCI 534 (2014) 86–100

- ★ *Boolean simulations and bisimulations for* ***max-plus automata***

Komenda, Lahaye, Boimond, IFAC PAPERS ONLINE 51-7 (2018) 192–197

# Weakly linear systems – for max-plus matrices

- ***How to extend WLS's to the context of max-plus matrices?***
  - How to compute general (non-Boolean) solutions?
- ***How to apply WLS's to max-plus automata?***
  - How to get general (non-Boolean) simulations and bisimulations?

- for FFAs
  - *general **fuzzy solutions** provide better state reductions than Boolean (crisp) solutions*
  - *there are cases when there exist general **fuzzy simulations and bisimulations** but there are no any Boolean (crisp) simulations and bisimulations*

- ★ **_Quantale_** – an algebra $\mathbb{Q} = (Q, \vee, \wedge, \otimes, 0, 1)$ where
  - (Q1) $(Q, \vee, \wedge, 0, 1)$ is a complete lattice
  - (Q2) $(Q, \otimes)$ is a semigroup
  - (Q3) infinite distributive laws are satisfied: $a \otimes \left(\bigvee_{s\in I} a_s\right) = \bigvee_{s\in I} (a \otimes a_s)$, $\left(\bigvee_{s\in I} a_s\right) \otimes a = \bigvee_{s\in I} (a_s \otimes a)$

  ➠ ***each quantale is residuated***

- ★ **_Unital quantale_** – $\mathbb{Q} = (Q, \vee, \wedge, \otimes, 0, 1, e)$, $(Q, \otimes, e)$ is a monoid with identity $e$ (in general $e \neq 1$)

- ★ **_max-plus algebra_** (also: ***max-plus semiring***) – $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \vee, +, -\infty, 0)$
- ★ **_max-plus quantale_** (also: ***complete max-plus algebra, complete max-plus semiring***) – ***commutative unital quantale*** $\mathbb{R}_{\infty} = (\mathbb{R} \cup \{-\infty, +\infty\}, \vee, \wedge, +, -\infty, +\infty, 0)$ where

$$a + b = \begin{cases} a + b & \text{if } a, b \in \mathbb{R}, \\ -\infty & \text{if } a = -\infty \text{ or } b = -\infty, \\ +\infty & \text{if } a = +\infty, b \neq -\infty \text{ or } b = +\infty, a \neq -\infty. \end{cases}$$

# Max-plus quantale – residuation

- ⋆ *max-plus algebra* **does not have residuation**, *but* **max-plus quantale does**
- ⋆ in the general case ☞ *two different residuum operations in a quantale*
- ⋆ due to commutativity ☞ *only one residuum operation in the max-plus quantale*

$$a \to b = \begin{cases} b - a & \text{if } a, b \in \mathbb{R}, \\ -\infty & \text{if } b = -\infty \text{ or } a \neq -\infty, \\ +\infty & \text{if } b = +\infty \text{ or } a = b = -\infty. \end{cases}$$

- ⋆ *residuum operations on matrices* ☞ $A \in \mathbb{R}_\infty^{m\times n}$, $B \in \mathbb{R}_\infty^{n\times k}$, $C \in \mathbb{R}_\infty^{m\times k}$

$$(A\backslash C)(j,l) = \bigwedge_{i=1}^{m} A(i,j) \to C(i,l), \qquad (C/B)(i,j) = \bigwedge_{l=1}^{k} B(j,l) \to C(i,l).$$

- ⋆ *residuation property:* $AB \leqslant C \;\Leftrightarrow\; A \leqslant C/B \;\Leftrightarrow\; B \leqslant A\backslash C$

# Weakly linear systems over $\mathbb{R}_\infty$ – the greatest solutions

Stamenković, Ćirić, Djurdjanović, DISCRETE EVENT DYNAMIC SYSTEMS (accepted)

**Theorem 1 (The greatest solution in $\mathbb{R}_\infty^{m\times n}$)**

*The set of all solutions of a WLS over $\mathbb{R}_\infty$ is a complete lattice.*

*Consequently, there exists the greatest solution of this WLS in $\mathbb{R}_\infty^{m\times n}$.*

**Theorem 2 (The greatest solutions in $\mathbb{R}_{\max}^{m\times n}$)**

*It is not necessary that there is the greatest solution of the WLS in $\mathbb{R}_{\max}^{m\times n}$.*

*However, for each $X_0 \in \mathbb{R}_{\max}^{m\times n}$ there exists the greatest solution of the WLS less than or equal to $X_0$.*

# Weakly linear systems over $\mathbb{R}_\infty$ – the equivalent form

★ $\phi : \mathbb{R}_\infty^{m\times n} \to R_\infty^{m\times n}$

$$\phi(C) = \left(\bigwedge_{i\in I_1} A_i\backslash(CB_i)\right) \wedge \left(\bigwedge_{i\in I_2} (A_iC)/B_i\right) \wedge \left(\bigwedge_{i\in I_3} ((B_iC^\top)/A_i)^\top\right) \wedge \left(\bigwedge_{i\in I_4} (B_i\backslash(C^\top A_i))^\top\right)$$

**Theorem 3 (The equivalent form)**

(1) *The WLS is equivalent to a single inequation $X \leqslant \phi(X)$.*

(2) *$\phi$ is isotone and $\omega$-continuous.*

★ *$\omega$-continuous* – preserves infima of non-increasing sequences in $\mathbb{R}_\infty^{m\times n}$

★ *solving WLS* ☞ *computing the greatest post-fixed point of $\phi$*

★ sequence of matrices ☞ given $X_0 \in \mathbb{R}_\infty^{m\times n}$ we define

$$X_1 = X_0, \qquad X_{n+1} = \phi(X_n) \wedge X_n, \qquad S(X_0) = \bigwedge_{n\in\mathbb{N}} X_n.$$

**Theorem 3 (Computing the greatest solution)**

*$S(X_0)$ is the greatest solution to $X \leqslant \phi(X)$, $X \leqslant X_0$.*

*(the greatest solution to the WLS contained in $X_0$)*

★ *this does not necessarily hold in the fuzzy context*

- **Problem:** *How to efficiently compute the greatest solution?*

- $S(X_0)$ is ***finitely computable*** if $\{X_n\}_{n\in\mathbb{N}}$ converges in a finite number of steps
- there is $n \in \mathbb{N}$ such that $X_{n+1} = X_n$ ☞ then $S(X_0) = X_n$
- ***number of computational steps*** ☞ the smallest such $n$

- $d : \mathbb{R}_\infty^{m\times n} \times R_\infty^{m\times n} \to \mathbb{R}^+ \cup \{0, +\infty\}$ ☞ ***extended metric***

$$d(A,B) = \begin{cases} \inf\{\lambda \in \mathbb{R}^+ \mid (-\lambda)B \leqslant A \leqslant B\} & \text{if } \{\lambda \in \mathbb{R}^+ \mid (-\lambda)B \leqslant A \leqslant B\} \neq \emptyset \\ +\infty & \text{otherwise} \end{cases}$$

$$d(A,B) = \max_{i,j}\{|A(i,j) - B(i,j)|\} \qquad \textit{(Chebyshev distance)}$$

★ ***finite solutions*** ☞ solutions that are ***finite matrices*** ☞ entries only from $\mathbb{R}$

**Theorem 4 (Finitely computable solutions)**

*Let the WLS has a finite solution $Y$ and suppose that there is $\lambda \in \mathbb{R}$ such that $\lambda X_0$, $\lambda A_i$'s and $\lambda B_i$'s take entries in $\mathbb{Z} \cup \{-\infty, +\infty\}$.*

*Then $S(X_0)$ is finitely computable and the corresponding number of computational steps does not exceed* $\mathbf{2}mn \cdot d(\lambda Y, \lambda X_0)$.

**Consequences**

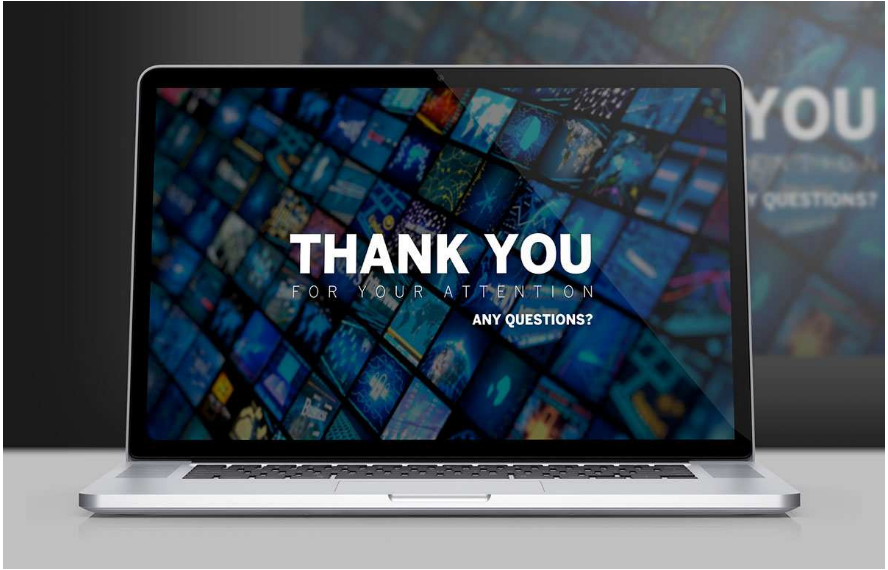
*(1) If $X_0$, $A_i$'s and $B_i$'s take entries in $\mathbb{Z} \cup \{-\infty, +\infty\}$, then $S(X_0)$ is finitely computable and the corresponding number of computational steps does not exceed* $\mathbf{2}mn \cdot d(Y, X_0)$.

*(2) If $X_0$, $A_i$'s and $B_i$'s take entries in $\mathbb{Q} \cup \{-\infty, +\infty\}$, then $S(X_0)$ is finitely computable and the corresponding number of computational steps does not exceed* $\mathbf{2}mn \cdot d(\lambda Y, \lambda X_0)$*, where $\lambda$ is the least common multiple of all entries of $X_0$, $A_i$'s and $B_i$'s.*

# Our ongoing research

- ⋆ *Applications of WLS's to* ***max-plus automata***
  - ➠ ***simulations and bisimulations for max-plus automata***
  - ➠ ***state reduction for max-plus automata***
  - ➠ ***determinization of max-plus automata***

- ⋆ for simulations and bisimulations
  - ➠ we expect much better algorithms than the existing ones, which are based on Boolean solutions to WLS's


THANK YOU
FOR YOUR ATTENTION
ANY QUESTIONS?