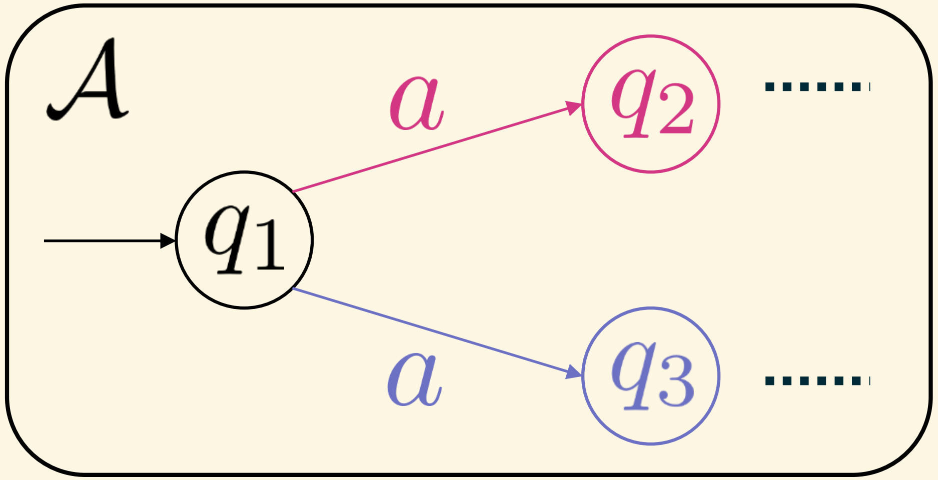
Disambiguation of Weighted Tree Automata

joint work with Markus Ulbricht

Kevin Stier

21.04.2021

Powerset Construction



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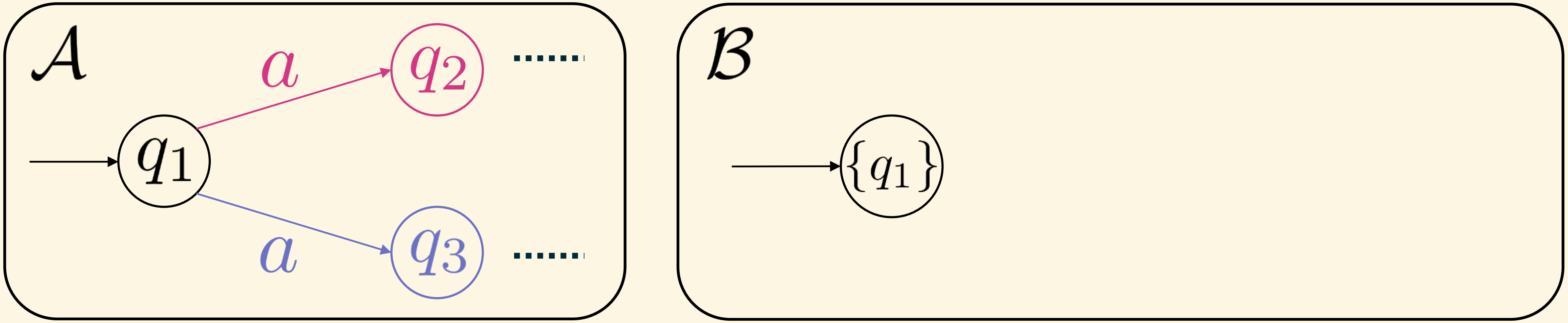

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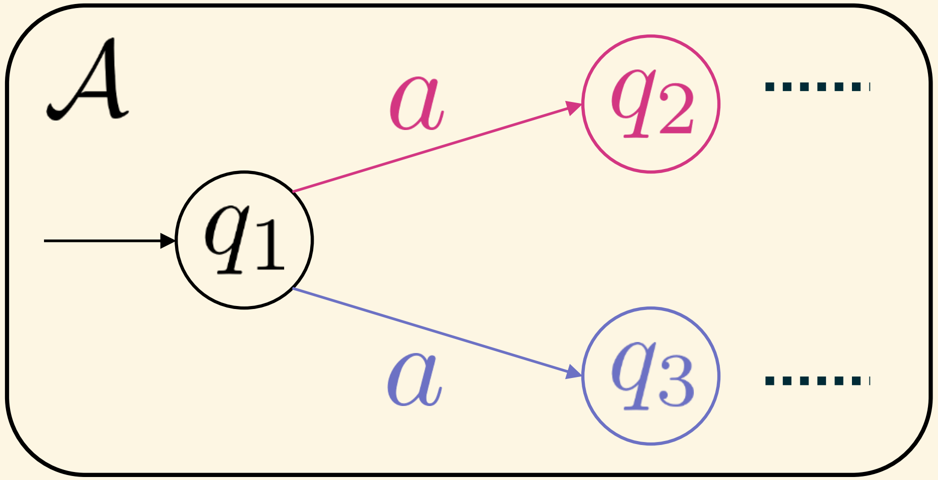

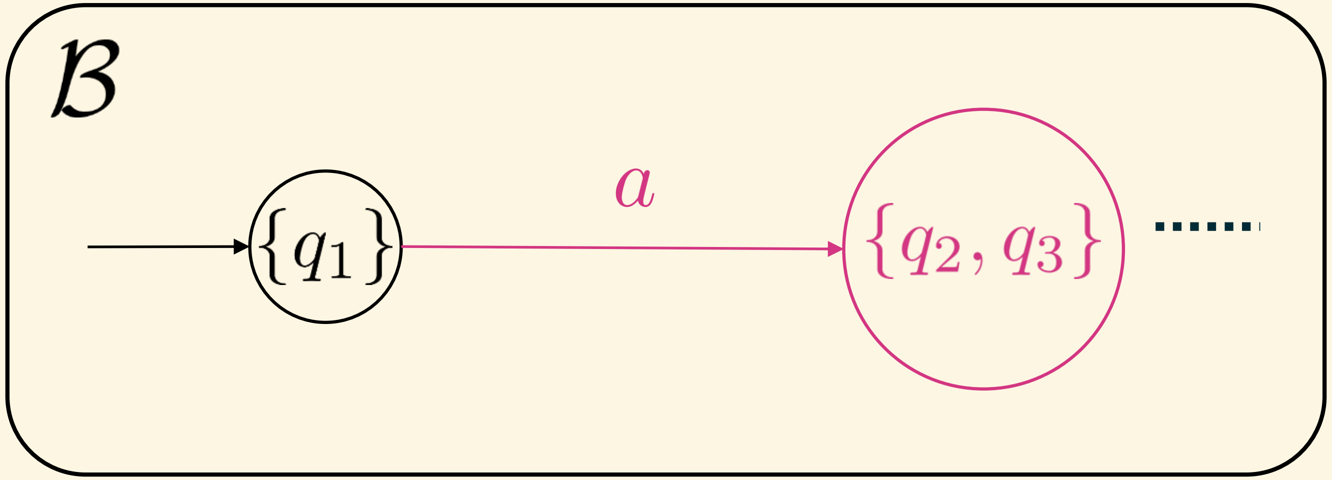

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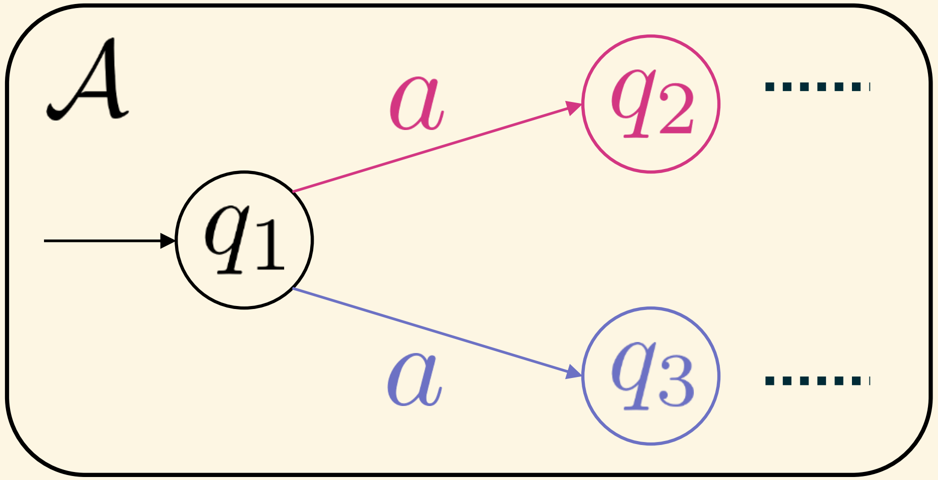
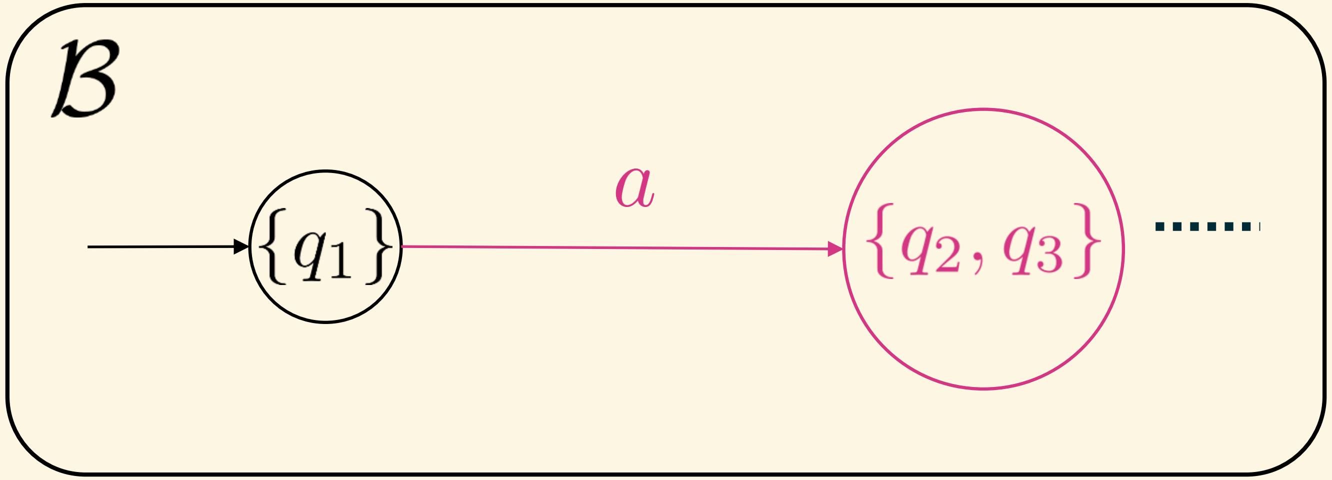
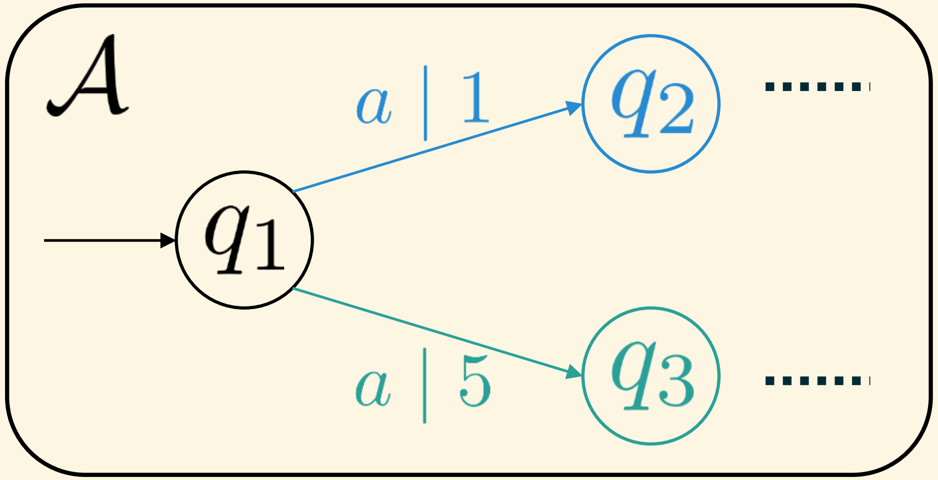




Factorization



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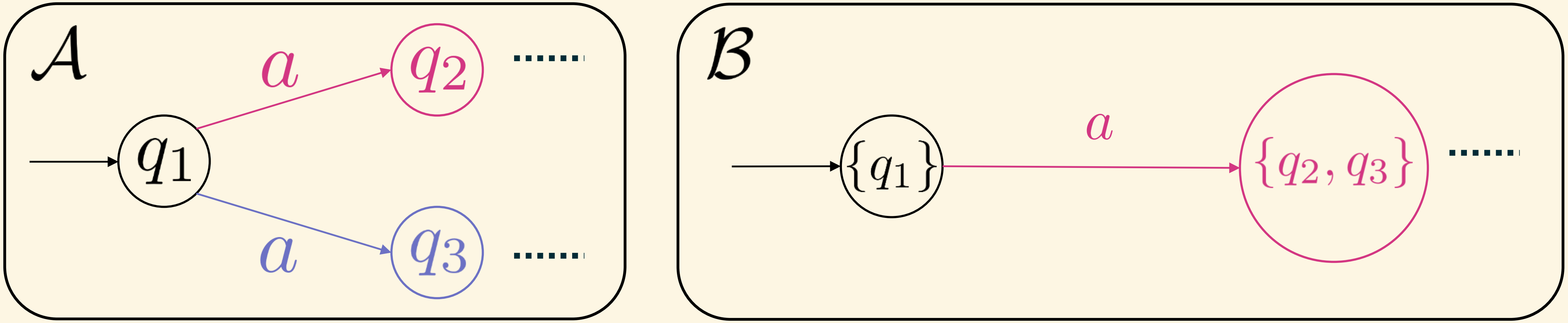

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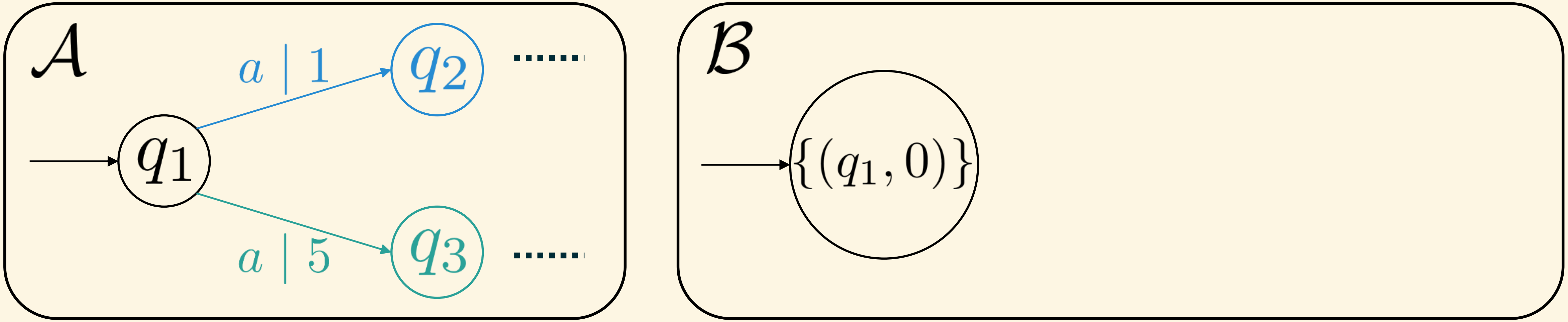

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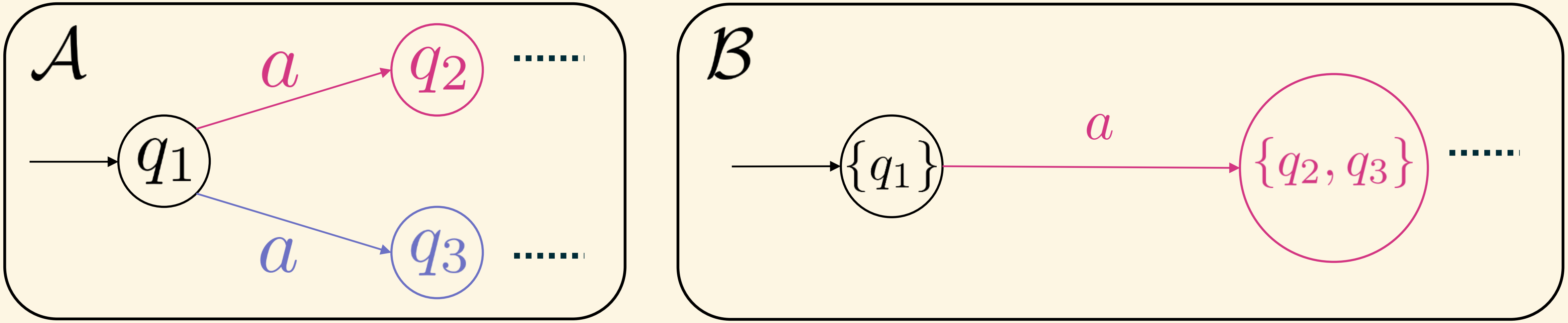

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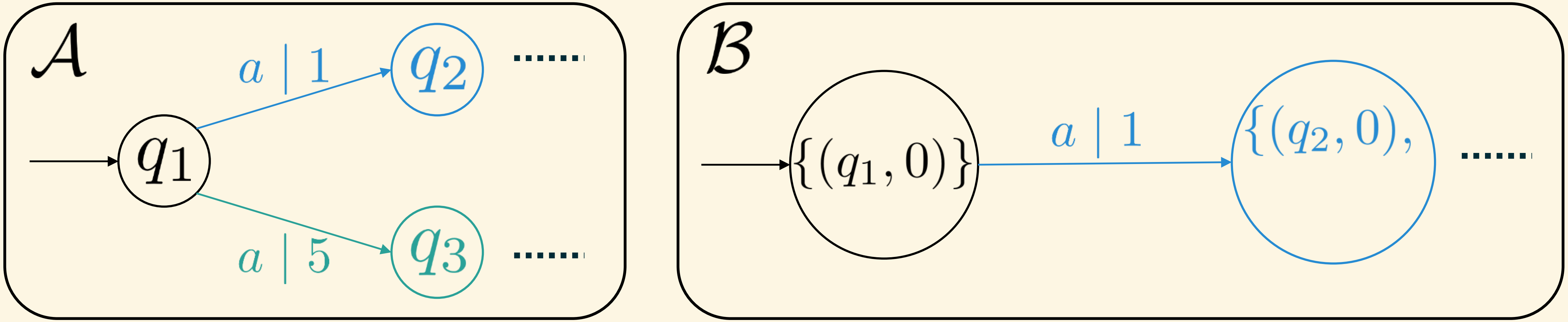

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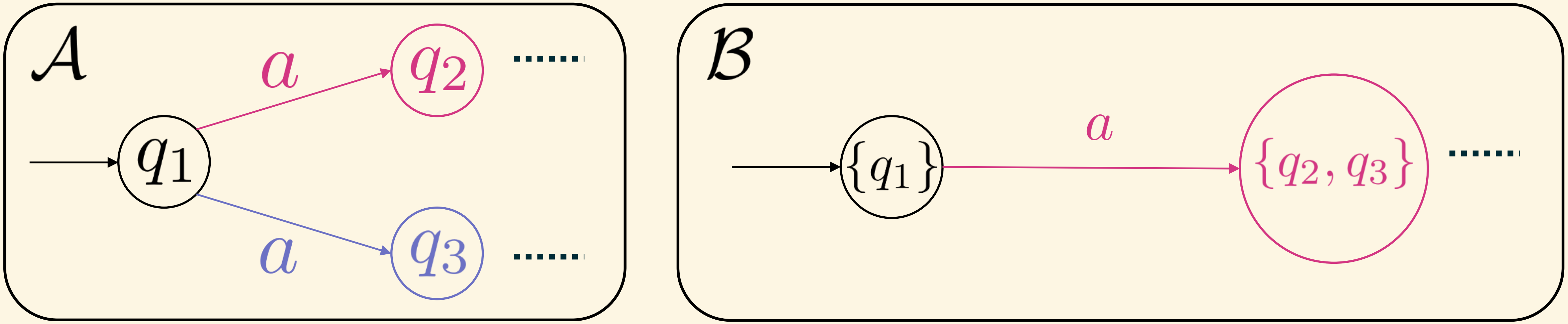

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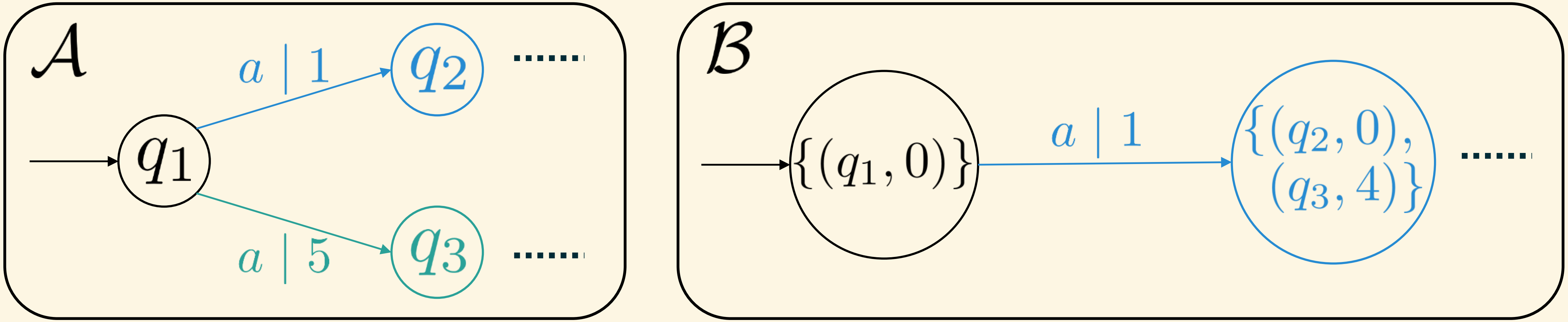
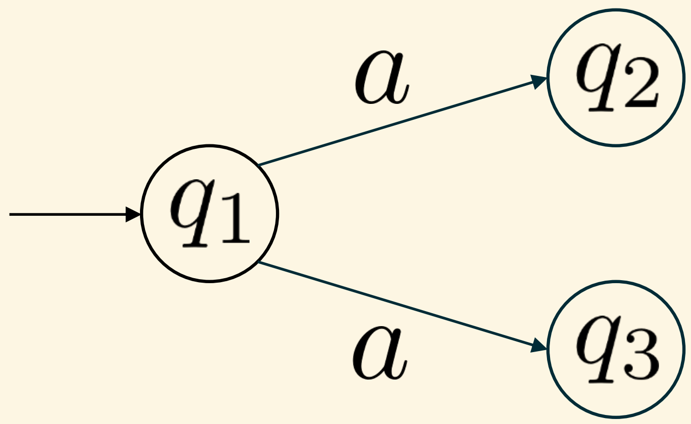


Determinism vs Unambiguity



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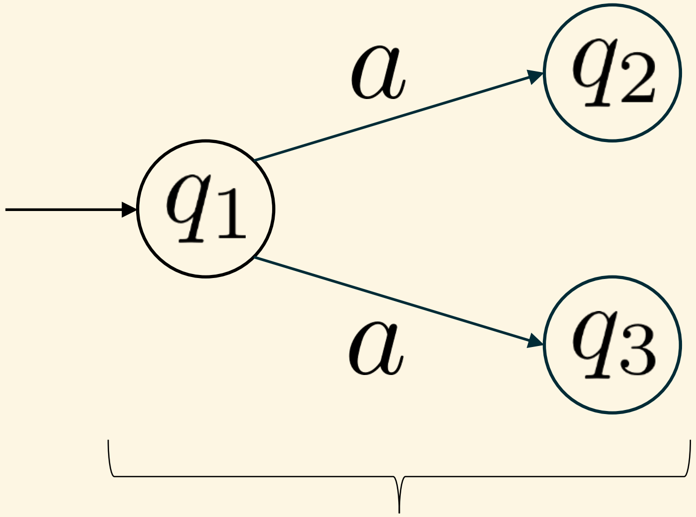
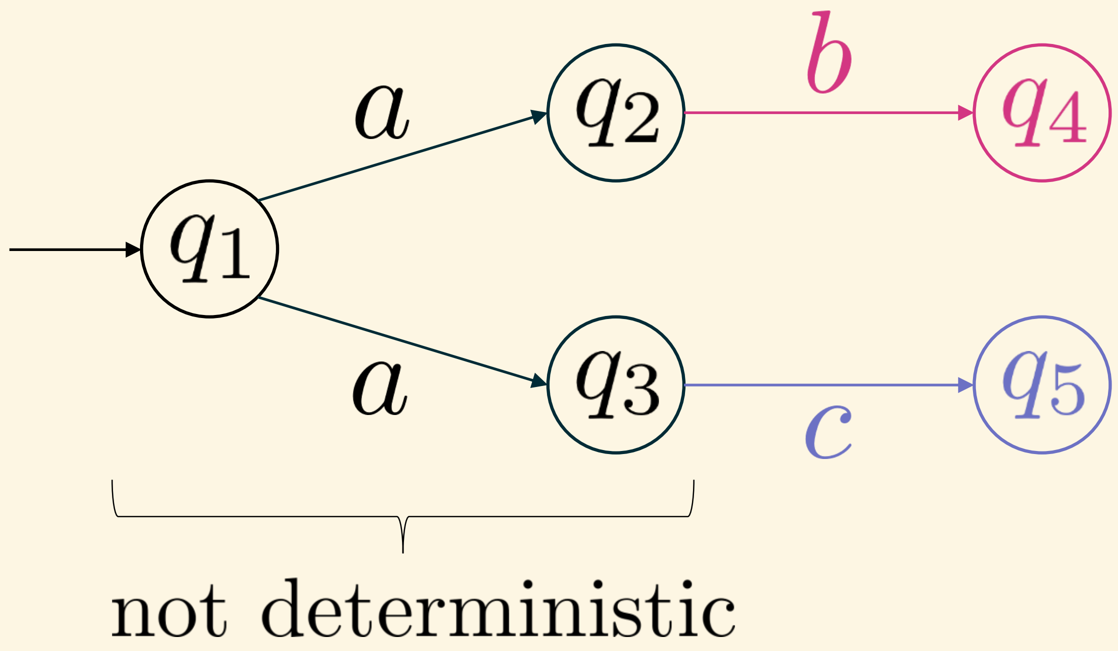


not deterministic

Determinism vs Unambiguity



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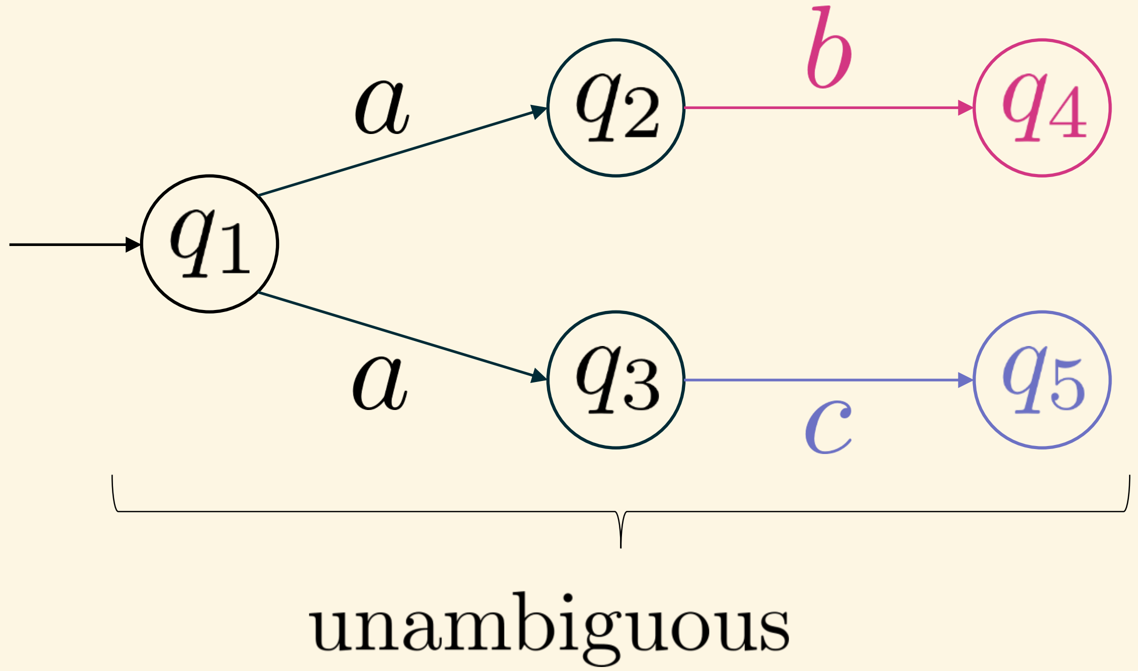

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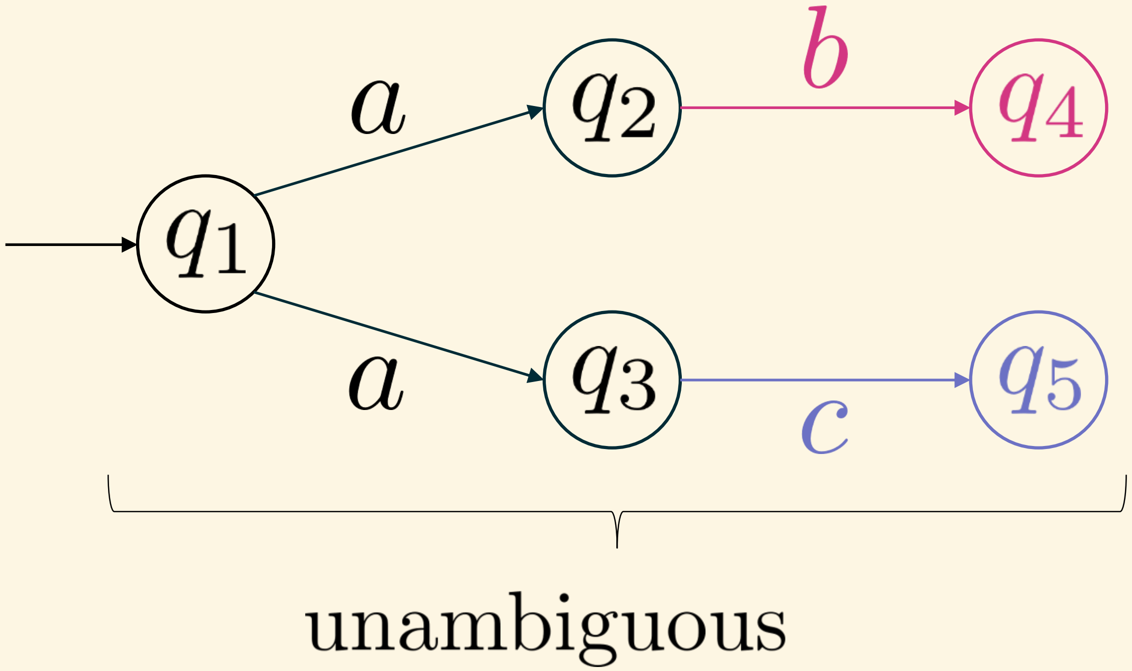
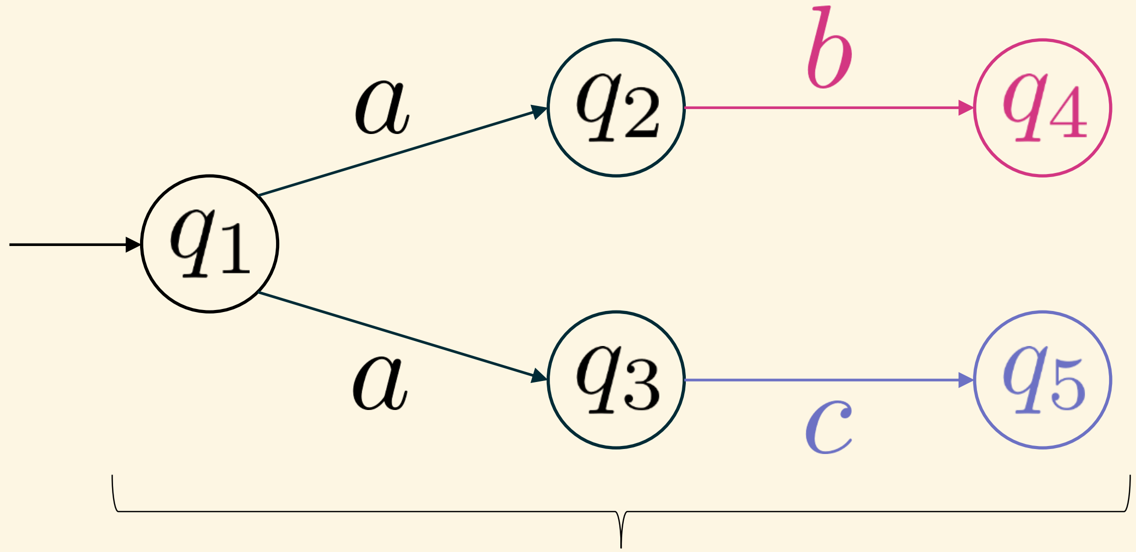


common future

Determinism vs Unambiguity



unambiguous

common future

A disambiguation algorithm for weighted automata

Mehryar Mohri and Michael D. Riley (2017)

(Theoretical Computer Science, 679, 53-68)

Overview

WTA $\mathcal{T}$

Overview

WTA $\mathcal{T}$

uniformity construction

WTA $\mathcal{U}$

Overview

WTA $\mathcal{T}$

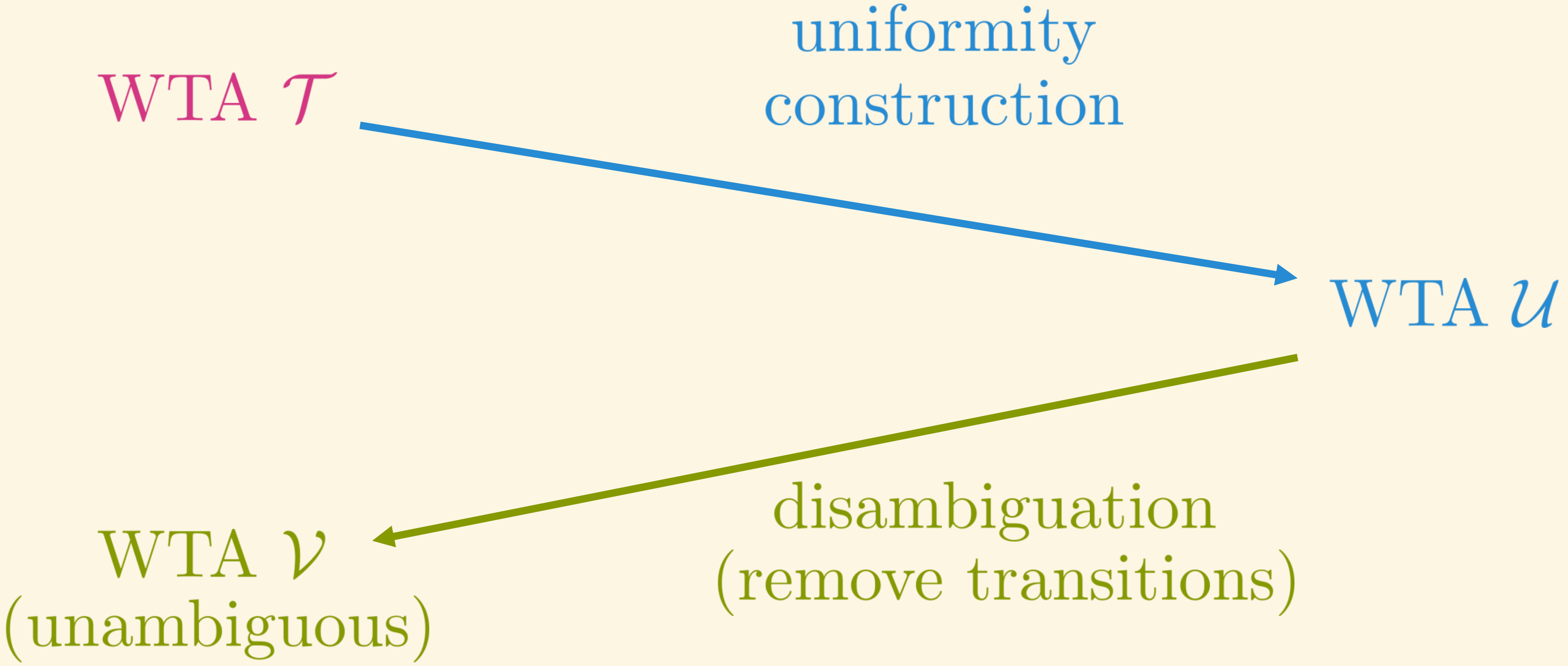
uniformity construction

WTA $\mathcal{U}$

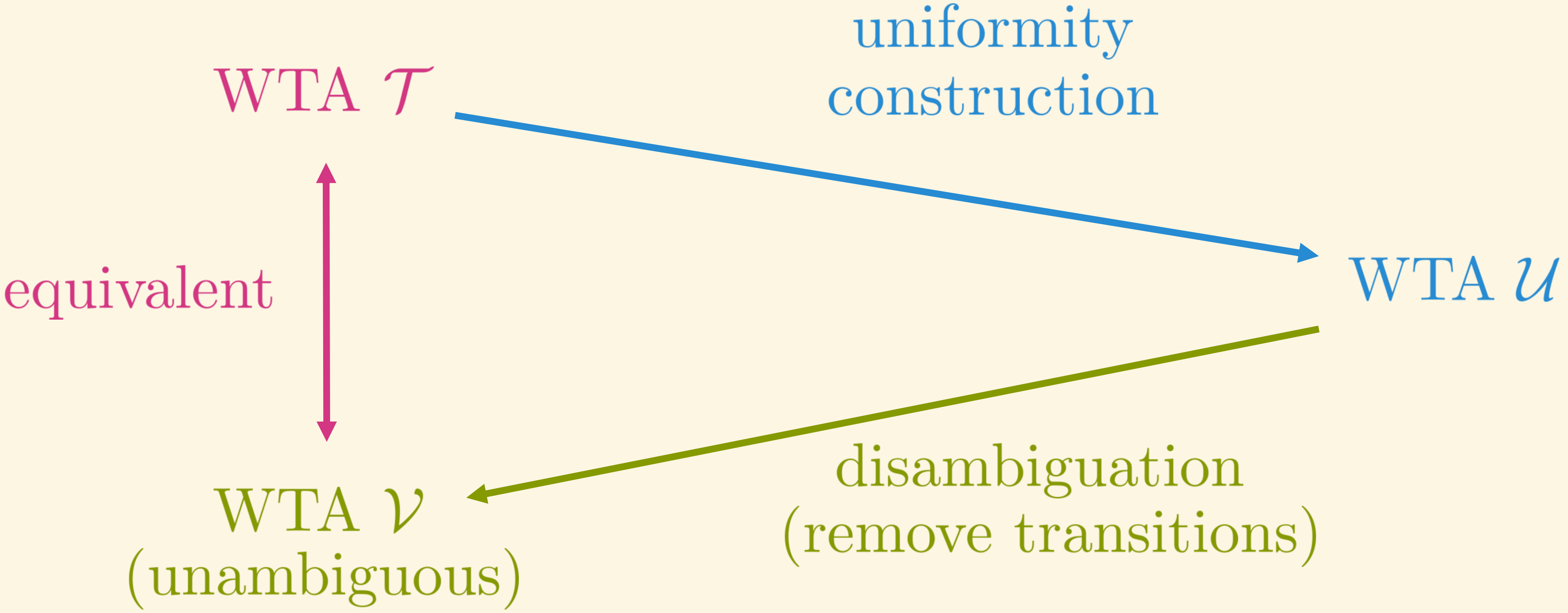
Theorem

$$\mathrm{wt}_{\mathcal{U}}(r) \otimes \nu_{\mathcal{U}}(r(\varepsilon)) = [\![\mathcal{T}]\!](t)$$

Overview


WTA $\mathcal{T}$
uniformity
construction
WTA $\mathcal{U}$
disambiguation
(remove transitions)
WTA $\mathcal{V}$
(unambiguous)

Overview



WTA $\mathcal{T}$
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Weighted Tree Automaton $\mathcal{T}$

$$(Q, \Sigma, S, \mu, \nu)$$

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finite set of states

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ranked alphabet

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tropical semiring

$(\mathbb{R}_\infty, \min, +, \infty, 0)$

+artic semiring

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transition weights
$(\mu_\sigma : Q^k \times Q \to S)_{k,\sigma}$

Weighted Tree Automaton $\mathcal{T}$

$$(Q, \Sigma, S, \mu, \nu)$$

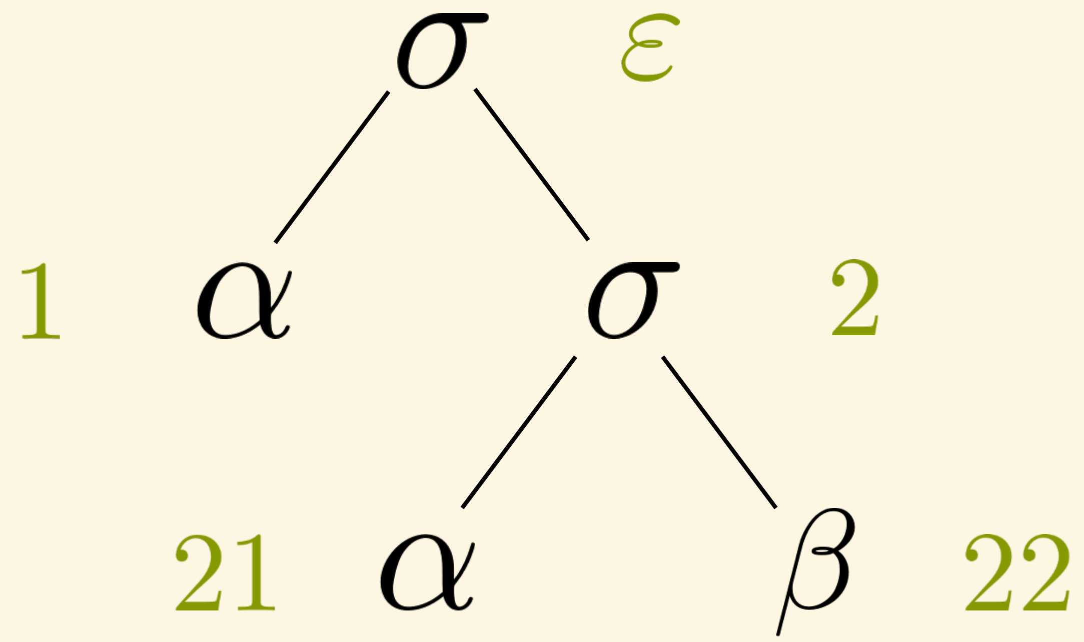
- Q: finite set of states
- Σ: ranked alphabet
- S: tropical semiring $(\mathbb{R}_\infty, \min, +, \infty, 0)$ +artic semiring
- μ: transition weights $(\mu_\sigma : Q^k \times Q \to S)_{k,\sigma}$
- ν: $\nu : Q \to S$

Positions

$\mathrm{pos}(t)$

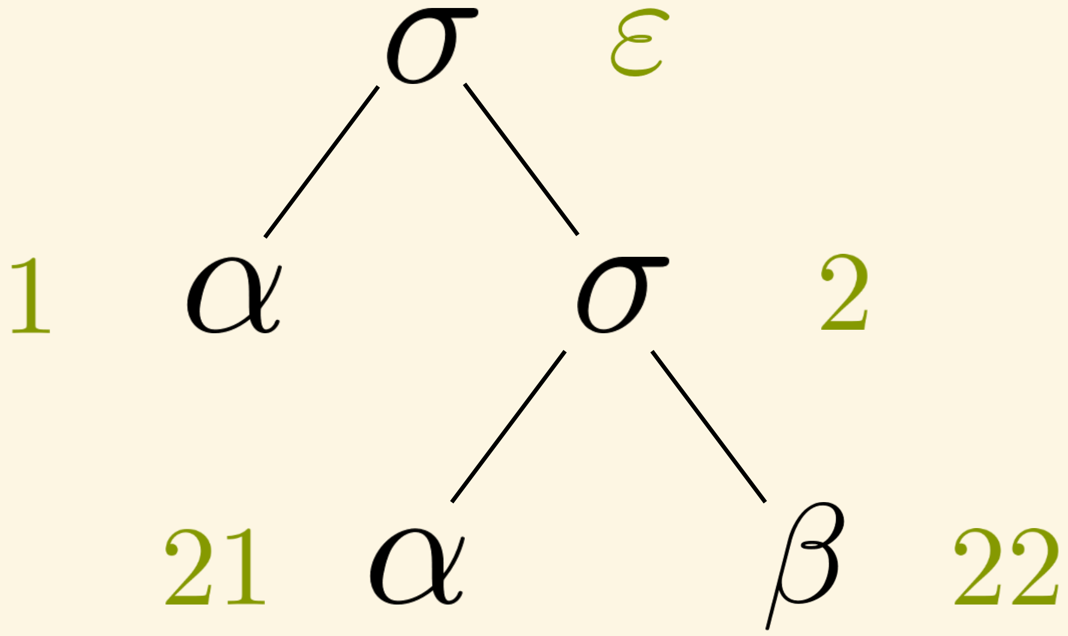
Positions

pos(t)



Runs

$r : \mathrm{pos}(t) \to Q$



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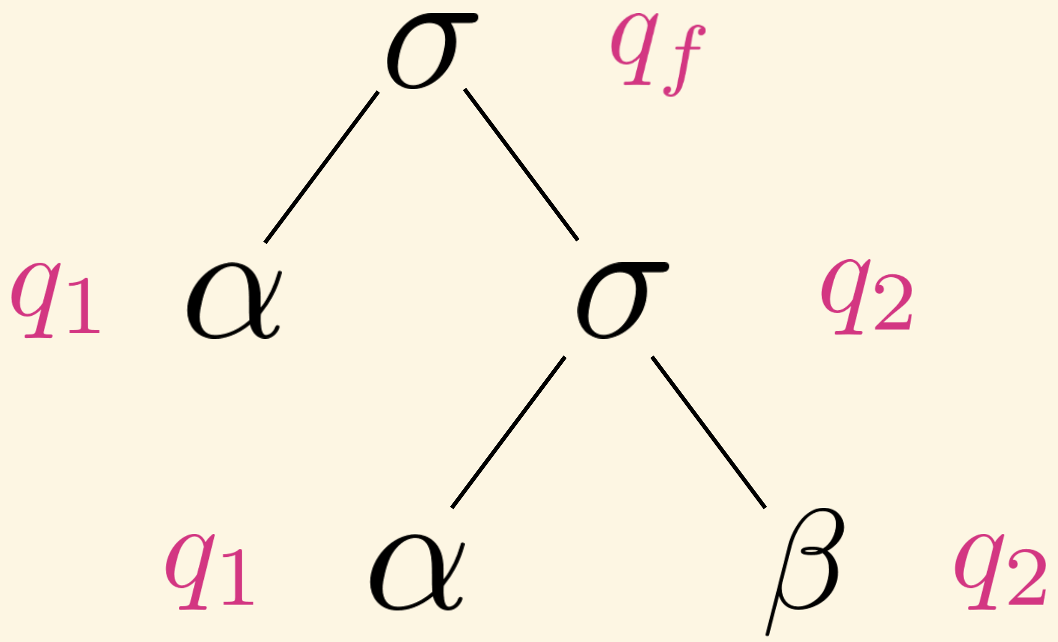

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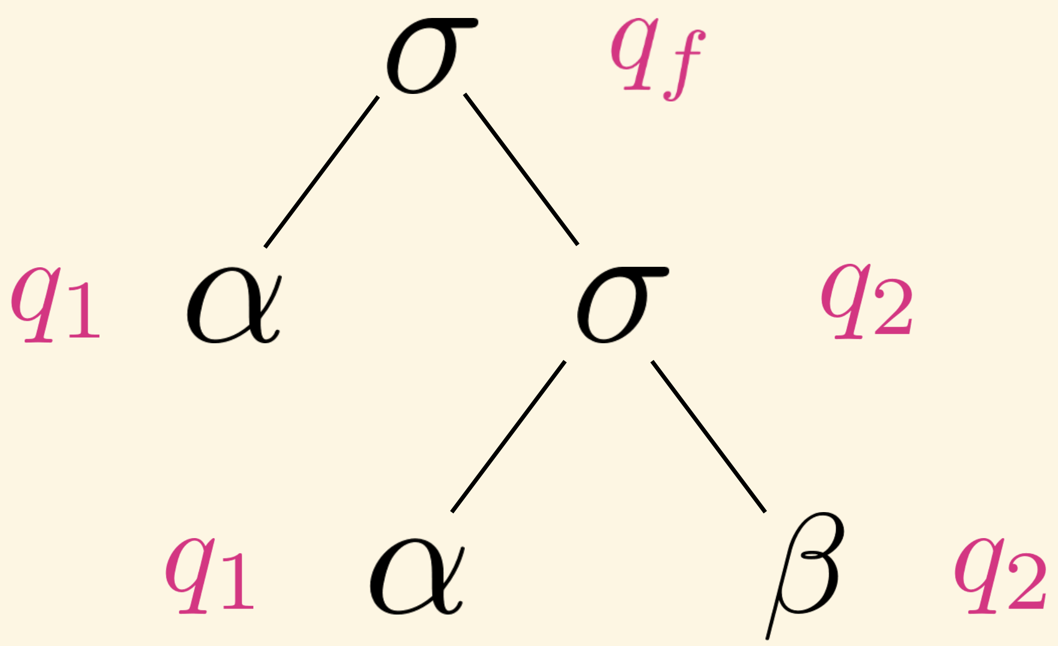
Set of runs

$\mathrm{Run}_{\mathcal{T}}(t)$



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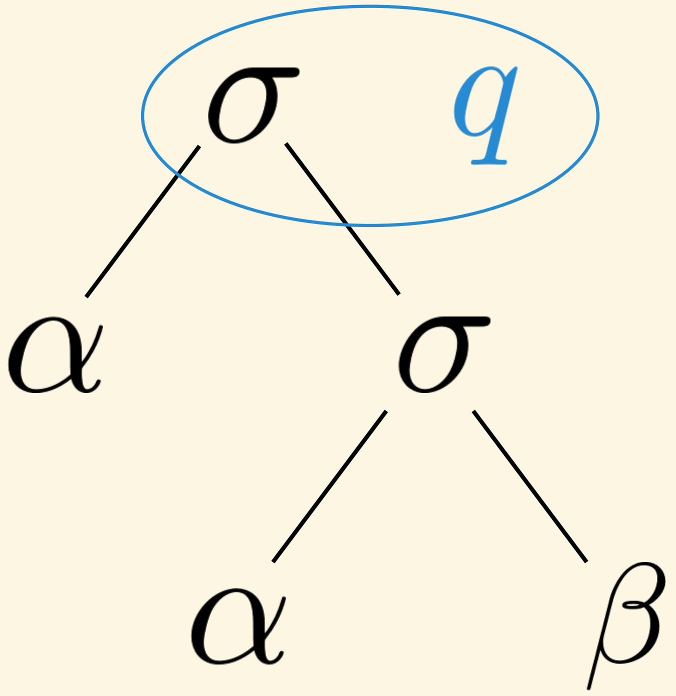

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$\mathrm{Run}_{\mathcal{T}}(t)$ $\mathrm{Run}_{\mathcal{T}}(p, C)$

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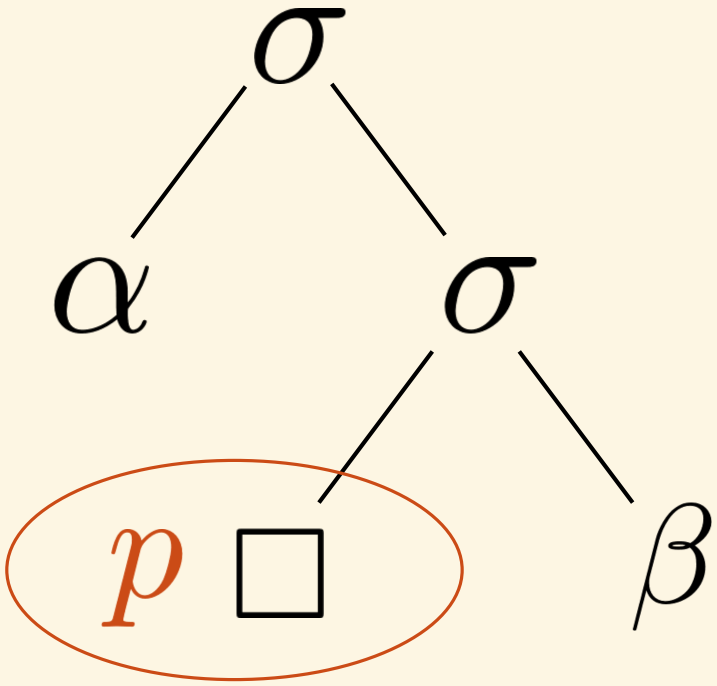

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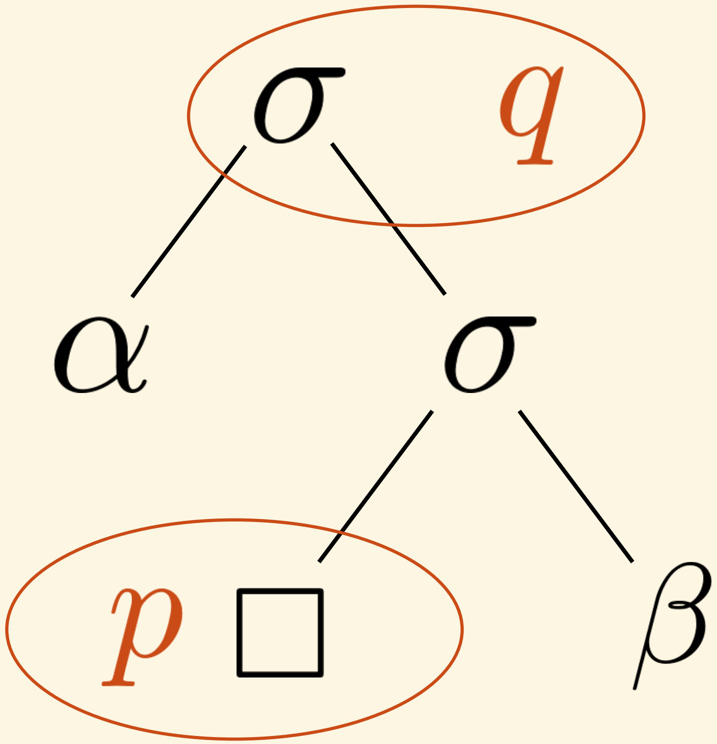

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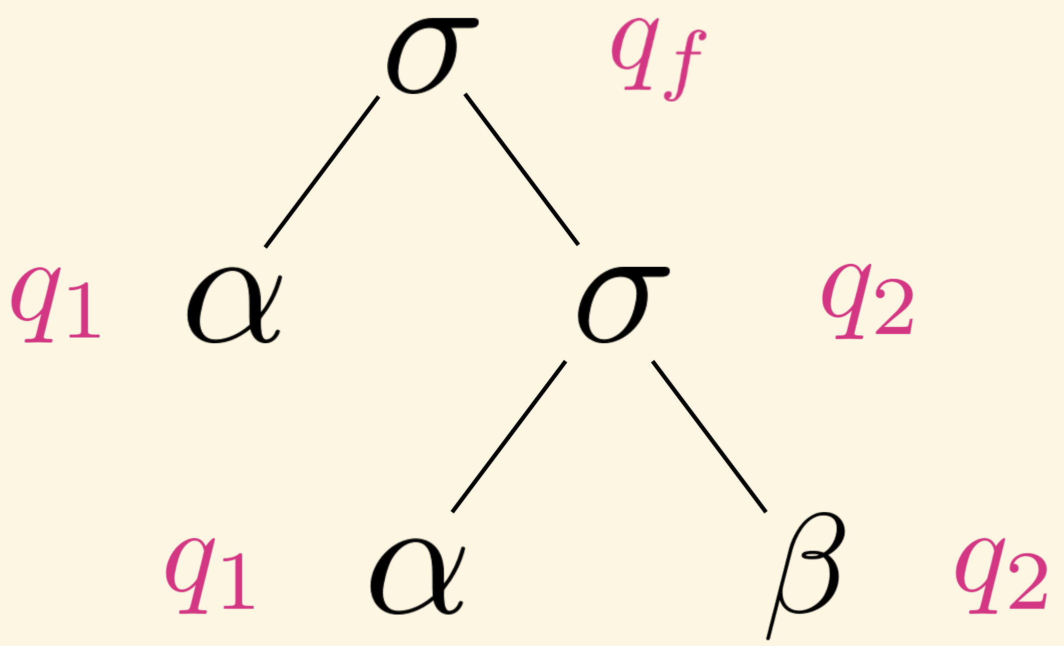

Weight of runs

$$\mathrm{wt}(r) = \mu(\alpha)(q_1) \otimes \mu(\beta)(q_2) \otimes \mu(\sigma)(q_1, q_2, q_2) \otimes \dots$$

Runs

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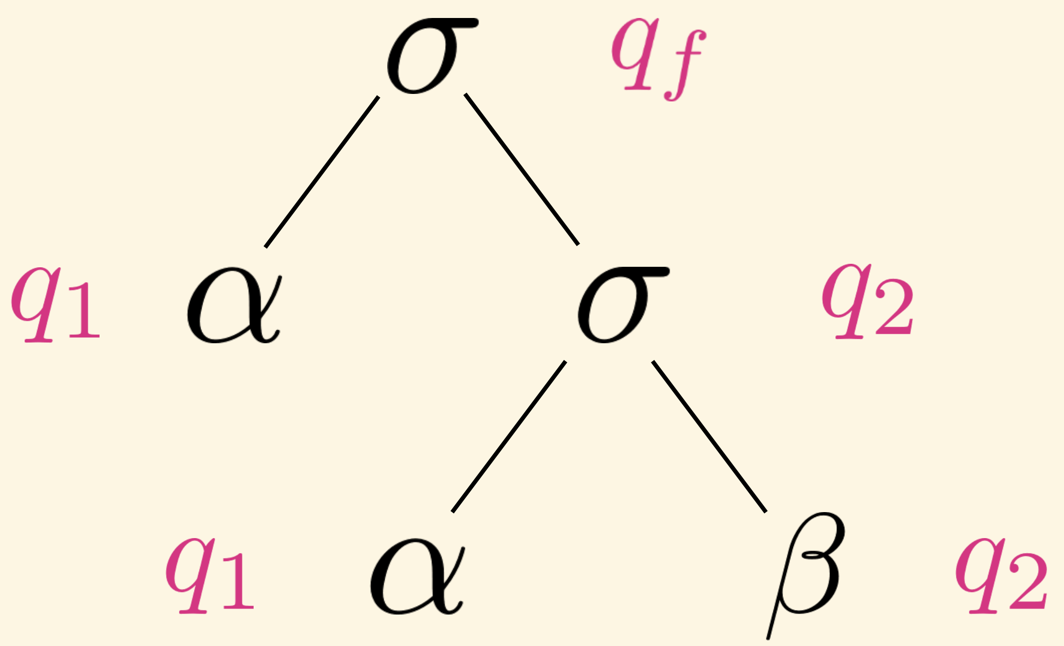

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Runs

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Running example $\mathcal{T}$ over $(\mathbb{R}^{\geq 0}, +, \cdot)$

$\Sigma = \{\alpha^{(0)}, \beta^{(0)}, \sigma^{(2)}\}$

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Runs on $\sigma(\alpha, \alpha)$

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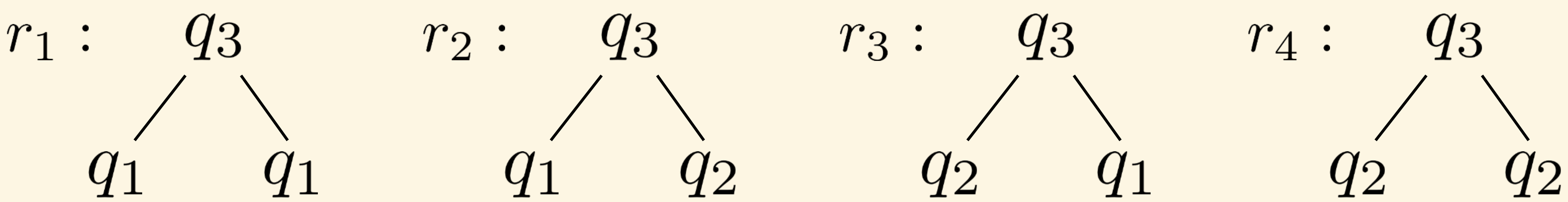

Running example $\mathcal{T}$ over $(\mathbb{R}^{\geq 0}, +, \cdot)$

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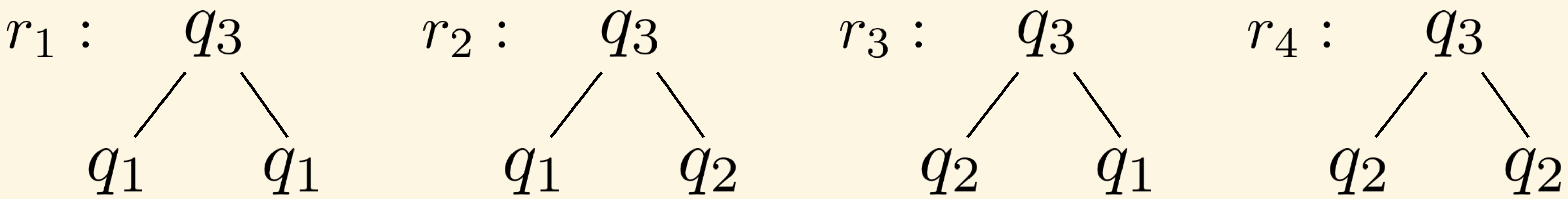

$wt_{\mathcal{T}}(r_1) = 45$ $\quad wt_{\mathcal{T}}(r_2) = 24$ $\quad wt_{\mathcal{T}}(r_3) = 24$ $\quad wt_{\mathcal{T}}(r_4) = 20$

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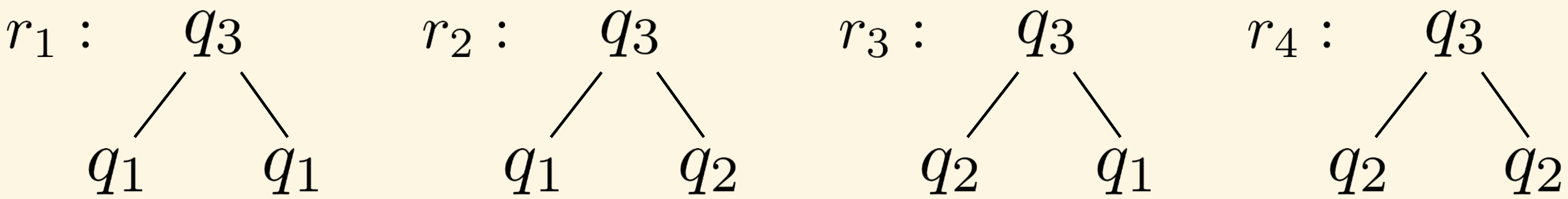
Runs on $\sigma(\alpha, \alpha)$ $\Rightarrow [\![\mathcal{T}]\!](\sigma(\alpha, \alpha)) = 113$



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Common future relation

$p\mathrm{R}q$

Common future relation

$p \mathrm{R} q$ iff $\exists C \in \mathrm{C}_\Sigma :$

Common future relation

$$p \mathrm{R} q \quad \text{iff} \quad \exists C \in \mathrm{C}_\Sigma : \quad [\![\mathcal{T}]\!](p, C) \neq 0, \quad [\![\mathcal{T}]\!](q, C) \neq 0$$

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$$\mathrm{R}(p) = \{q \in Q \mid p\mathrm{R}q\}$$

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Uniformity construction

$$u(t, p)_q = \left\{ \right.$$

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Uniformity construction

$$u(t,p)_q = \begin{cases} & \text{iff } q \in \mathrm{R}(p) \\ 0 & \text{otherwise} \end{cases}$$

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$$Q_{\mathcal{U}} = \{u(t,p) \in S^Q \mid t \in \mathrm{T}_\Sigma, p \in Q_{\mathcal{T}}\}$$

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$$Q_{\mathcal{U}} = \{u(t,p) \in S^{Q} \mid t \in \mathrm{T}_{\Sigma}, p \in Q_{\mathcal{T}}\}$$

For $u(t_1,p_1),\ldots,u(t_k,p_k),u(t,p) \in Q_{\mathcal{U}}$, $t = \sigma(t_1,\ldots,t_k)$

Uniformity construction

$$Q_{\mathcal{U}} = \{u(t,p) \in S^Q \mid t \in \mathrm{T}_\Sigma, p \in Q_{\mathcal{T}}\}$$

For $u(t_1, p_1), \ldots, u(t_k, p_k), u(t,p) \in Q_{\mathcal{U}}$, $t = \sigma(t_1, \ldots, t_k)$

$$\mu_{\mathcal{U}}(\sigma)(u(t_1,p_1), \ldots, u(t_k,p_k), u(t,p)) = \begin{cases} w & \text{iff (1) and (2) holds} \\ 0 & \text{otherwise} \end{cases}$$

Uniformity construction

$$Q_{\mathcal{U}} = \{u(t,p) \in S^Q \mid t \in \mathrm{T}_\Sigma, p \in Q_{\mathcal{T}}\}$$

For $u(t_1,p_1),\dots,u(t_k,p_k),u(t,p) \in Q_{\mathcal{U}}$, $t = \sigma(t_1,\dots,t_k)$

$$\mu_{\mathcal{U}}(\sigma)(u(t_1,p_1),\dots,u(t_k,p_k),u(t,p)) = \begin{cases} w & \text{iff (1) and (2) holds} \\ 0 & \text{otherwise} \end{cases}$$

$$w = \bigoplus_{(q_1,\dots,q_k)\in Q^k} u(t_1,p_1)_{q_1} \otimes \cdots \otimes u(t_k,p_k)_{q_k} \otimes \bigoplus_{q \in \mathrm{supp}(u(t,p))} \mu_{\mathcal{T}}(\sigma)(q_1,\dots,q_k,q)$$

Uniformity construction

$$Q_{\mathcal{U}} = \{u(t,p) \in S^Q \mid t \in \mathrm{T}_\Sigma, p \in Q_{\mathcal{T}}\}$$

For $u(t_1, p_1), \ldots, u(t_k, p_k), u(t, p) \in Q_{\mathcal{U}}$, $t = \sigma(t_1, \ldots, t_k)$

$$\mu_{\mathcal{U}}(\sigma)(u(t_1,p_1), \ldots, u(t_k,p_k), u(t,p)) = \begin{cases} w & \text{iff (1) and (2) holds} \\ 0 & \text{otherwise} \end{cases}$$

---

(1) $\mu_{\mathcal{T}}(\sigma)(p_1, \ldots, p_k, p) \neq 0$

Uniformity construction

$$Q_{\mathcal{U}} = \{u(t,p) \in S^Q \mid t \in \mathrm{T}_\Sigma, p \in Q_{\mathcal{T}}\}$$

For $u(t_1,p_1),\ldots,u(t_k,p_k),u(t,p) \in Q_{\mathcal{U}}$, $t = \sigma(t_1,\ldots,t_k)$

$$\mu_{\mathcal{U}}(\sigma)(u(t_1,p_1),\ldots,u(t_k,p_k),u(t,p)) = \begin{cases} w & \text{iff (1) and (2) holds} \\ 0 & \text{otherwise} \end{cases}$$

---

(1) $\mu_{\mathcal{T}}(\sigma)(p_1,\ldots,p_k,p) \neq 0$

(2) $u(t,p)_q = w^{-1} \otimes \bigoplus_{(q_1,\ldots,q_k)\in Q^k} u(t_1,p_1)_{q_1} \otimes \cdots \otimes u(t_k,p_k)_{q_k} \otimes \mu_{\mathcal{T}}(\sigma)(q_1,\ldots,q_k,q)$

Uniformity construction

$$Q_{\mathcal{U}} = \{u(t,p) \in S^Q \mid t \in \mathrm{T}_\Sigma, p \in Q_{\mathcal{T}}\}$$

For $u(t_1,p_1),\ldots,u(t_k,p_k),u(t,p) \in Q_{\mathcal{U}},\ t = \sigma(t_1,\ldots,t_k)$

$$\mu_{\mathcal{U}}(\sigma)(u(t_1,p_1),\ldots,u(t_k,p_k),u(t,p)) = \begin{cases} w & \text{iff (1) and (2) holds} \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_{\mathcal{U}}(u(t,p)) = \bigoplus_{q \in Q} u(t,p)_q \otimes \nu_{\mathcal{T}}(q)$$

Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$

Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

$$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$$

$$u_1 = u(\alpha, q_1) = \begin{pmatrix} 3/5 \\ 2/5 \\ 0 \end{pmatrix}$$

Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

$$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$$

$$u_1 = u(\alpha, q_1) = \begin{pmatrix} 3/5 \\ 2/5 \\ 0 \end{pmatrix}$$

$$u_2 = u(\alpha, q_2) = \begin{pmatrix} 3/5 \\ 2/5 \\ 0 \end{pmatrix}$$

$$u_3 = u(\beta, q_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$u_4 = u(\sigma(\alpha, \alpha), q_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$ $\Delta_{\mathcal{U}} = \{\dots\}$

↓

$$\{\alpha \xrightarrow{5} u_1,\quad \beta \xrightarrow{1} u_3,\quad \sigma(u_3, u_3) \xrightarrow{5} u_4,\quad \sigma(x, y) \xrightarrow{113/25} u_4,$$

$$\alpha \xrightarrow{5} u_2,\quad \sigma(u_3, x) \xrightarrow{23/5} u_4,\quad \sigma(x, u_3) \xrightarrow{23/5} u_4,$$

$$\text{for } x, y \in \{u_1, u_2\}\}$$

Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$ $\quad \Delta_{\mathcal{U}} = \{\dots\}$ $\quad \nu_{\mathcal{U}} = (0, 0, 0, 1)$

$$\downarrow$$

$$\Big\{\alpha \xrightarrow{5} u_1, \quad \beta \xrightarrow{1} u_3, \quad \sigma(u_3, u_3) \xrightarrow{5} u_4, \quad \sigma(x, y) \xrightarrow{113/25} u_4,$$

$$\alpha \xrightarrow{5} u_2, \quad \sigma(u_3, x) \xrightarrow{23/5} u_4, \quad \sigma(x, u_3) \xrightarrow{23/5} u_4,$$

$$\text{for } x, y \in \{u_1, u_2\}\Big\}$$

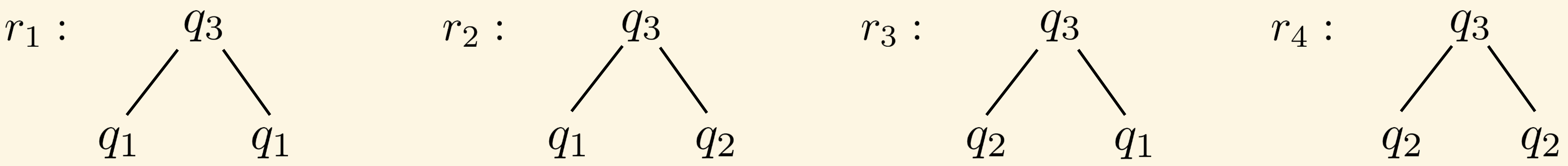
Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$ $\Delta_{\mathcal{U}} = \{\dots\}$ $\nu_{\mathcal{U}} = (0, 0, 0, 1)$

Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$ $\Delta_{\mathcal{U}} = \{\dots\}$ $\nu_{\mathcal{U}} = (0, 0, 0, 1)$

Runs of $\mathcal{T}$ on $\sigma(\alpha, \alpha)$

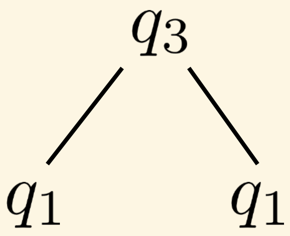


$wt_{\mathcal{T}}(r_1) = 45$ $wt_{\mathcal{T}}(r_2) = 24$ $wt_{\mathcal{T}}(r_3) = 24$ $wt_{\mathcal{T}}(r_4) = 20$

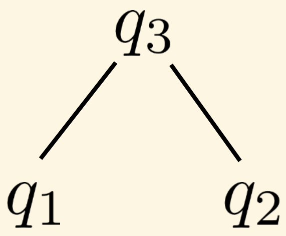
Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

$Q_\mathcal{U} = \{u_1, u_2, u_3, u_4\}$ $\Delta_\mathcal{U} = \{\dots\}$ $\nu_\mathcal{U} = (0, 0, 0, 1)$

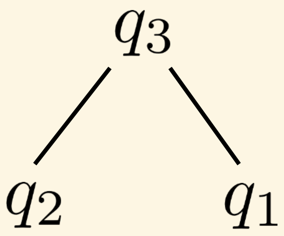
Runs of $\mathcal{T}$ on $\sigma(\alpha, \alpha)$ $\Rightarrow [\![\mathcal{T}]\!](\sigma(\alpha, \alpha)) = 113$

r_1 :


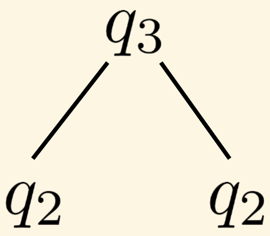
$wt_\mathcal{T}(r_1) = 45$

r_2 :


$wt_\mathcal{T}(r_2) = 24$

r_3 :


$wt_\mathcal{T}(r_3) = 24$

r_4 :


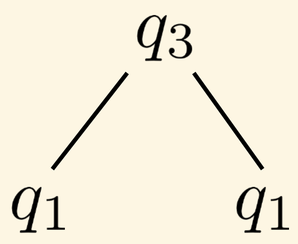
$wt_\mathcal{T}(r_4) = 20$

Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$ $\Delta_{\mathcal{U}} = \{\dots\}$ $\nu_{\mathcal{U}} = (0, 0, 0, 1)$

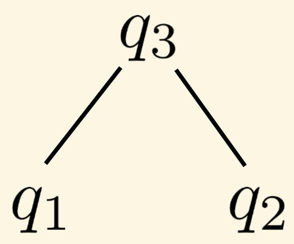
Runs of $\mathcal{T}$ on $\sigma(\alpha, \alpha)$ $\Rightarrow [\![\mathcal{T}]\!](\sigma(\alpha, \alpha)) = 113$

r_1 :



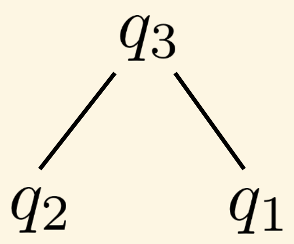
$wt_{\mathcal{T}}(r_1) = 45$

r_2 :



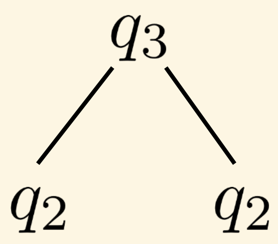
$wt_{\mathcal{T}}(r_2) = 24$

r_3 :



$wt_{\mathcal{T}}(r_3) = 24$

r_4 :



$wt_{\mathcal{T}}(r_4) = 20$

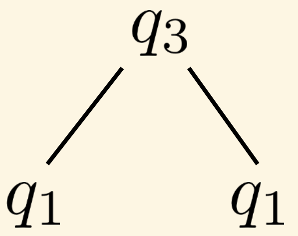
Runs of $\mathcal{U}$ on $\sigma(\alpha, \alpha)$

Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$ $\Delta_{\mathcal{U}} = \{\dots\}$ $\nu_{\mathcal{U}} = (0, 0, 0, 1)$

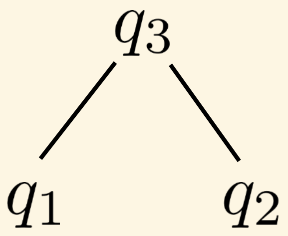
Runs of $\mathcal{T}$ on $\sigma(\alpha, \alpha)$ $\Rightarrow [\![\mathcal{T}]\!](\sigma(\alpha, \alpha)) = 113$

r_1 :



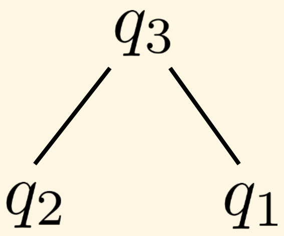
$wt_{\mathcal{T}}(r_1) = 45$

r_2 :



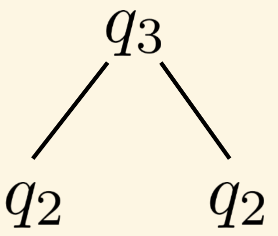
$wt_{\mathcal{T}}(r_2) = 24$

r_3 :



$wt_{\mathcal{T}}(r_3) = 24$

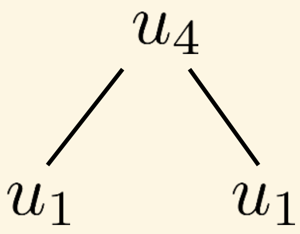
r_4 :



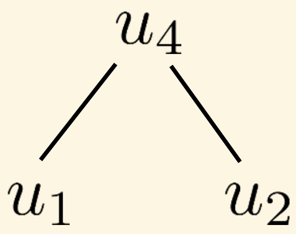
$wt_{\mathcal{T}}(r_4) = 20$

Runs of $\mathcal{U}$ on $\sigma(\alpha, \alpha)$

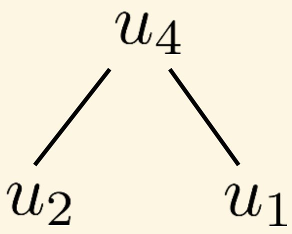
r'_1 :



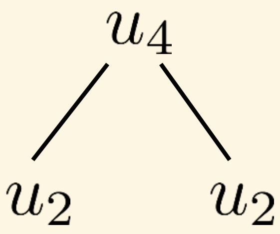
r'_2 :



r'_3 :



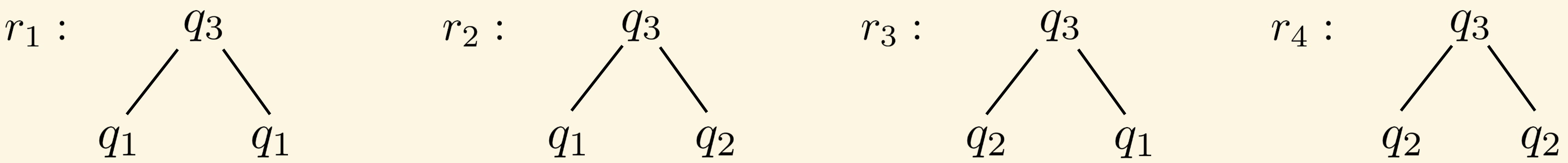
r'_4 :



Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

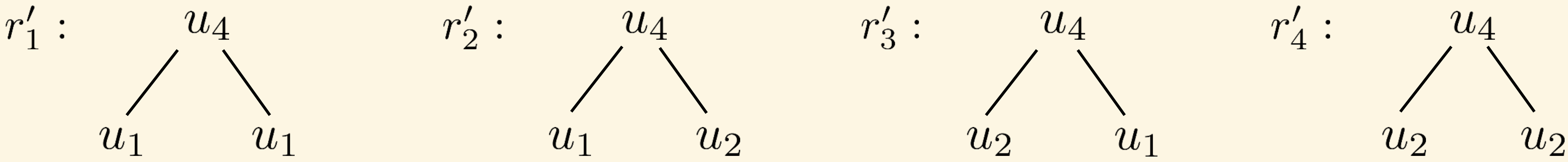
$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$ $\quad \Delta_{\mathcal{U}} = \{\dots\}$ $\quad \nu_{\mathcal{U}} = (0, 0, 0, 1)$

Runs of $\mathcal{T}$ on $\sigma(\alpha, \alpha)$ $\quad \Rightarrow [\![\mathcal{T}]\!](\sigma(\alpha, \alpha)) = 113$



$wt_{\mathcal{T}}(r_1) = 45$ $\quad wt_{\mathcal{T}}(r_2) = 24$ $\quad wt_{\mathcal{T}}(r_3) = 24$ $\quad wt_{\mathcal{T}}(r_4) = 20$

Runs of $\mathcal{U}$ on $\sigma(\alpha, \alpha)$



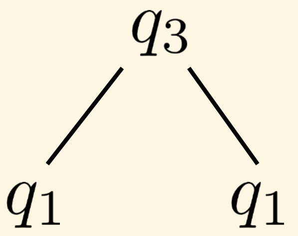
$wt_{\mathcal{T}}(r'_1) = 113$ $\quad wt_{\mathcal{T}}(r'_2) = 113$ $\quad wt_{\mathcal{T}}(r'_3) = 113$ $\quad wt_{\mathcal{T}}(r'_4) = 113$

Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$ $\Delta_{\mathcal{U}} = \{\dots\}$ $\nu_{\mathcal{U}} = (0, 0, 0, 1)$

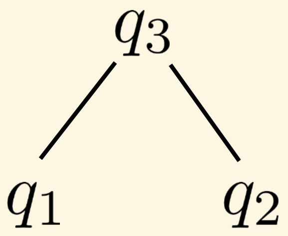
Runs of $\mathcal{T}$ on $\sigma(\alpha, \alpha)$ $\Rightarrow [\![\mathcal{T}]\!](\sigma(\alpha, \alpha)) = 113$

r_1 :



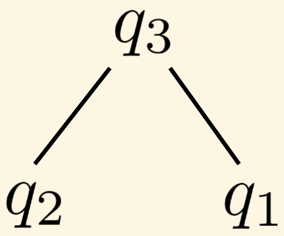
$wt_{\mathcal{T}}(r_1) = 45$

r_2 :



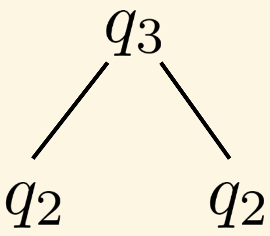
$wt_{\mathcal{T}}(r_2) = 24$

r_3 :



$wt_{\mathcal{T}}(r_3) = 24$

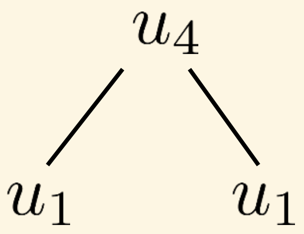
r_4 :



$wt_{\mathcal{T}}(r_4) = 20$

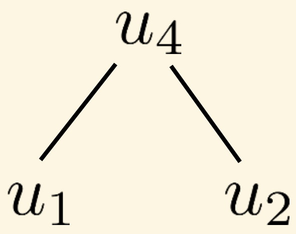
Runs of $\mathcal{U}$ on $\sigma(\alpha, \alpha)$ $\Rightarrow [\![\mathcal{U}]\!](\sigma(\alpha, \alpha)) = 4 \cdot 113$

r'_1 :



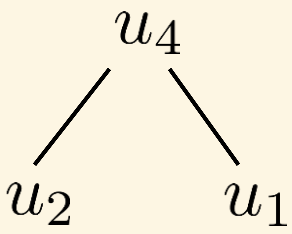
$wt_{\mathcal{T}}(r'_1) = 113$

r'_2 :



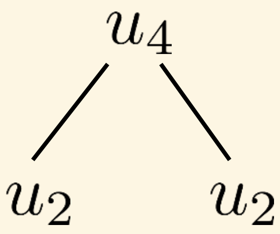
$wt_{\mathcal{T}}(r'_2) = 113$

r'_3 :



$wt_{\mathcal{T}}(r'_3) = 113$

r'_4 :

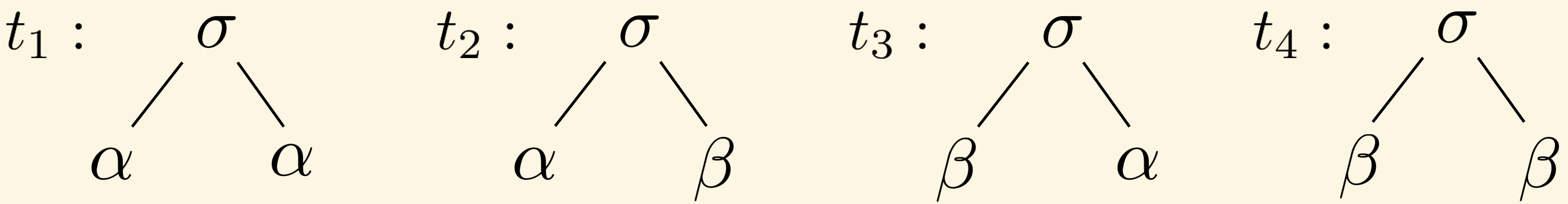


$wt_{\mathcal{T}}(r'_4) = 113$

Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$ $\Delta_{\mathcal{U}} = \{\dots\}$ $\nu_{\mathcal{U}} = (0, 0, 0, 1)$

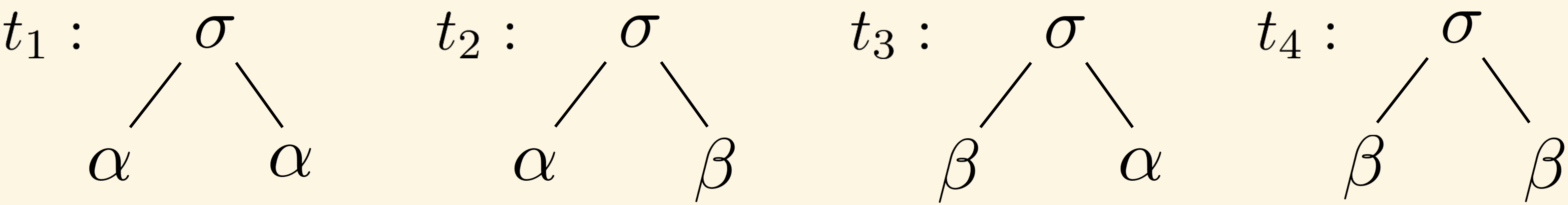
Accepted trees $\mathcal{T}$ and $\mathcal{U}$



Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$ $\Delta_{\mathcal{U}} = \{\dots\}$ $\nu_{\mathcal{U}} = (0, 0, 0, 1)$

Accepted trees $\mathcal{T}$ and $\mathcal{U}$



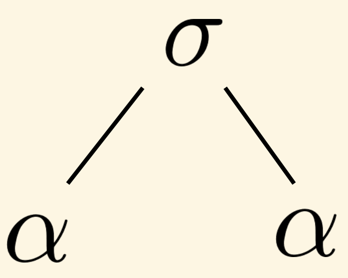
$[\![\mathcal{T}]\!](t_1) = 113$ $[\![\mathcal{T}]\!](t_1) = 69$ $[\![\mathcal{T}]\!](t_1) = 69$ $[\![\mathcal{T}]\!](t_1) = 5$

Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

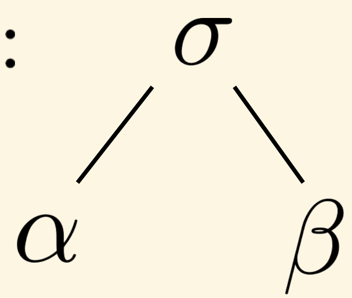
$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$ $\Delta_{\mathcal{U}} = \{\dots\}$ $\nu_{\mathcal{U}} = (0, 0, 0, 1)$

Accepted trees $\mathcal{T}$ and $\mathcal{U}$

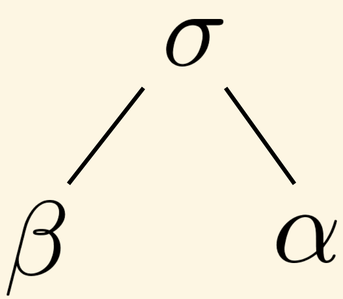
$t_1 :$



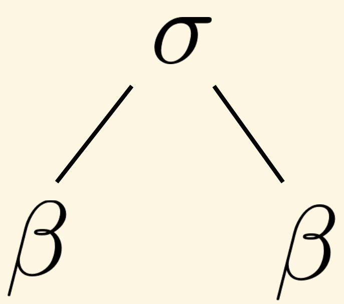
$t_2 :$



$t_3 :$



$t_4 :$



$[\![\mathcal{T}]\!](t_1) = 113$ $[\![\mathcal{T}]\!](t_1) = 69$ $[\![\mathcal{T}]\!](t_1) = 69$ $[\![\mathcal{T}]\!](t_1) = 5$

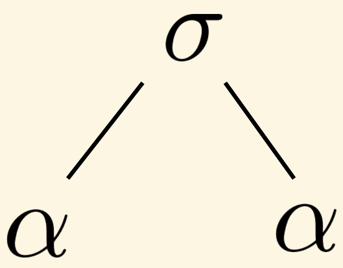
$[\![\mathcal{U}]\!](t_1) = 4 \cdot 113$

Running example (uniformity: $\mathcal{T} \to \mathcal{U}$)

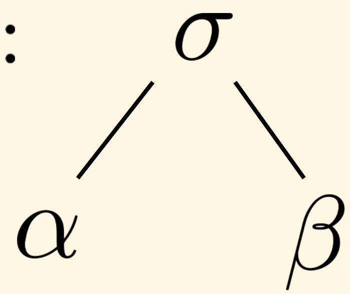
$Q_{\mathcal{U}} = \{u_1, u_2, u_3, u_4\}$ $\Delta_{\mathcal{U}} = \{\dots\}$ $\nu_{\mathcal{U}} = (0, 0, 0, 1)$

Accepted trees $\mathcal{T}$ and $\mathcal{U}$

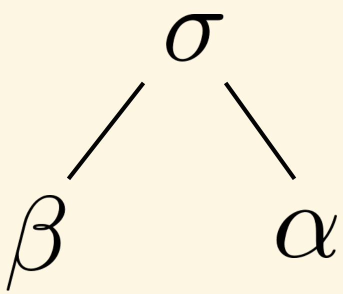
t_1 :



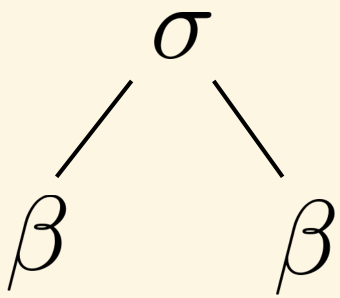
t_2 :



t_3 :



t_4 :



$[\![\mathcal{T}]\!](t_1) = 113$ $[\![\mathcal{T}]\!](t_1) = 69$ $[\![\mathcal{T}]\!](t_1) = 69$ $[\![\mathcal{T}]\!](t_1) = 5$

$[\![\mathcal{U}]\!](t_1) = 4 \cdot 113$ $[\![\mathcal{U}]\!](t_1) = 2 \cdot 69$ $[\![\mathcal{U}]\!](t_1) = 2 \cdot 69$ $[\![\mathcal{U}]\!](t_1) = 5$

Finiteness uniformity

Finiteness uniformity

Twins property

$\forall t \in \mathrm{T}_\Sigma, C \in \mathrm{C}_\Sigma, p, q \in Q$

Finiteness uniformity

Twins property

$\forall t \in \mathrm{T}_\Sigma, C \in \mathrm{C}_\Sigma, p, q \in Q$

$$\mathrm{wt}(\mathrm{Run}(t, q)) \neq 0, \mathrm{wt}(\mathrm{Run}(q, C, q)) \neq 0$$

$$\mathrm{wt}(\mathrm{Run}(t, p)) \neq 0, \mathrm{wt}(\mathrm{Run}(p, C, p)) \neq 0$$

Finiteness uniformity

Twins property

$\forall t \in \mathrm{T}_\Sigma, C \in \mathrm{C}_\Sigma, p, q \in Q$

$$\begin{matrix} \mathrm{wt}(\mathrm{Run}(t, q)) \neq 0, \mathrm{wt}(\mathrm{Run}(q, C, q)) \neq 0 \\ \mathrm{wt}(\mathrm{Run}(t, p)) \neq 0, \mathrm{wt}(\mathrm{Run}(p, C, p)) \neq 0 \end{matrix} \quad \Longrightarrow \quad \mathrm{wt}(\mathrm{Run}(p, C, p)) = \mathrm{wt}(\mathrm{Run}(q, C, q))$$

Finiteness uniformity

R-twins property

$\forall t \in \mathrm{T}_\Sigma, C \in \mathrm{C}_\Sigma, p, q \in Q$

$$\begin{array}{l} \mathrm{wt}(\mathrm{Run}(t,q)) \neq 0, \mathrm{wt}(\mathrm{Run}(q,C,q)) \neq 0 \\ \mathrm{wt}(\mathrm{Run}(t,p)) \neq 0, \mathrm{wt}(\mathrm{Run}(p,C,p)) \neq 0 \end{array} \implies \mathrm{wt}(\mathrm{Run}(p,C,p)) = \mathrm{wt}(\mathrm{Run}(q,C,q))$$

Finiteness uniformity

R-twins property

$\forall t \in \mathrm{T}_\Sigma, C \in \mathrm{C}_\Sigma, p, q \in Q$ with $p\mathrm{R}q$

$$\begin{array}{l} \mathrm{wt}(\mathrm{Run}(t, q)) \neq 0, \mathrm{wt}(\mathrm{Run}(q, C, q)) \neq 0 \\ \mathrm{wt}(\mathrm{Run}(t, p)) \neq 0, \mathrm{wt}(\mathrm{Run}(p, C, p)) \neq 0 \end{array} \implies \mathrm{wt}(\mathrm{Run}(p, C, p)) = \mathrm{wt}(\mathrm{Run}(q, C, q))$$

Finiteness uniformity

R-twins property

$\forall t \in \mathrm{T}_\Sigma, C \in \mathrm{C}_\Sigma, p, q \in Q$ with $p\mathrm{R}q$

$$\begin{matrix} \mathrm{wt}(\mathrm{Run}(t,q)) \neq 0, \mathrm{wt}(\mathrm{Run}(q,C,q)) \neq 0 \\ \mathrm{wt}(\mathrm{Run}(t,p)) \neq 0, \mathrm{wt}(\mathrm{Run}(p,C,p)) \neq 0 \end{matrix} \implies \mathrm{wt}(\mathrm{Run}(p,C,p)) = \mathrm{wt}(\mathrm{Run}(q,C,q))$$

Theorem

$\mathcal{T}$ over $S \in \{\mathbb{T}, \mathbb{A}\}$ + R-twins property $\implies$ $Q_{\mathcal{U}}$ finite

Thank you!