On the Universality Problem for Unambiguous Register Automata

joint work with Wojtek Czerwinski & Antoine Mottet

Karin Quaas
Universität Leipzig

- Extension of finite automata to infinite alphabets ($\mathbb{N}$)

- An accepting run on the input word 2 5 2 5 3 5:

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- Emptiness problem is decidable (PSPACE-complete)

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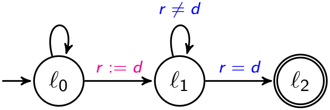

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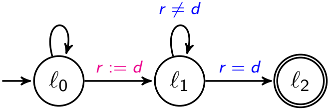

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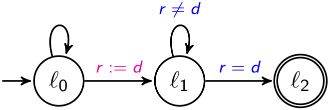

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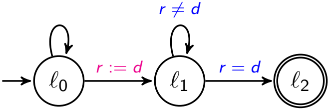

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- DETerministic $\subseteq$ UNAmbiguous $\subseteq$ NONDETerministic
- Collapses or non-collapses depending on model of computation
- UNA automata are often more succinct than DET automata
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- For register automata we have DET $\subsetneq$ UNA
- This follows from the non-closure of UNA under complement:
- $L = \{d_1 \dots d_k \mid \exists 1 \leq i < k . d_i = d_k\}$

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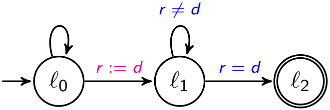

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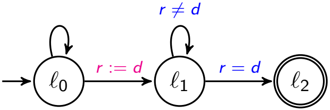

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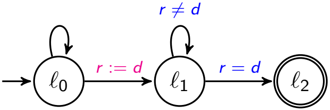

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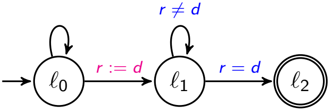

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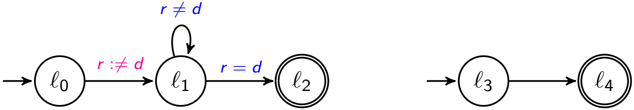

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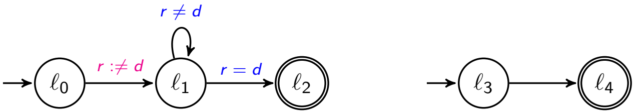

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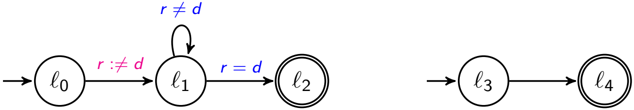

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- Universality Problem:
  given automaton $\mathcal{A}$, decide if $L(\mathcal{A}) = \mathbb{N}^*$
- State of the art in 2018:
- For register automata

# registers	DET	UNA	NONDET
1	PSPACE-complete	?	Ackermann-complete
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- Recall: the equivalence $L(\mathcal{A}) = \mathbb{N}^*$ iff $\overline{L(\mathcal{A})} \neq \emptyset$ cannot be used
- Table shows the upper bounds for universality (PSPACE-hardness follows from DET)

Unambiguous register automata	
without guessing	with guessing
2EXPSPACE [Mottet & Q, STACS 2019]	

- Reachability analysis on bounded graph (on-the-fly determinization)
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On-The-Fly Determinization

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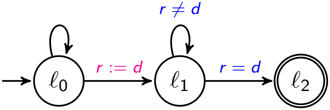

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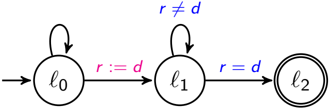

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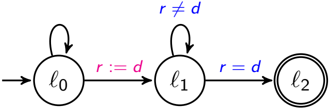

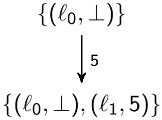

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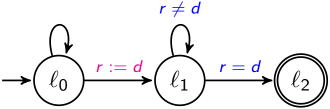

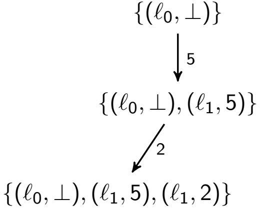

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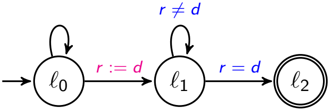

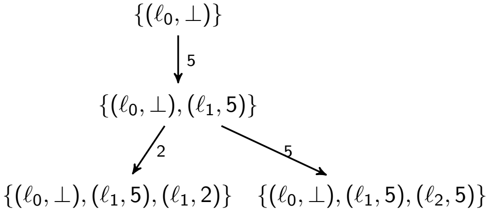

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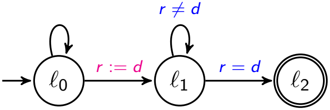

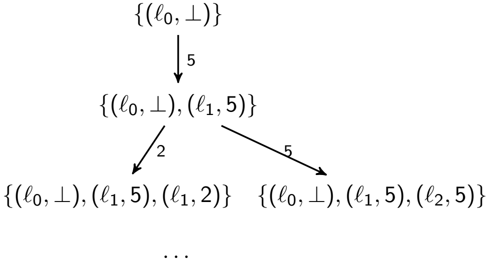

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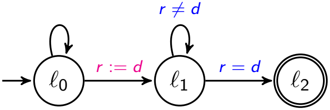

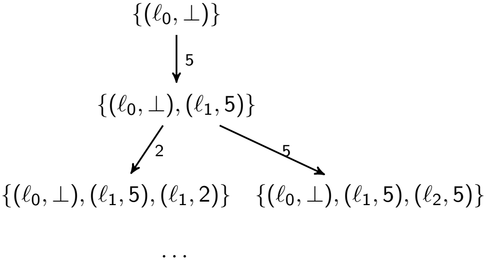

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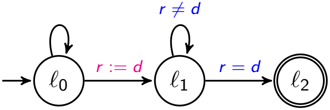

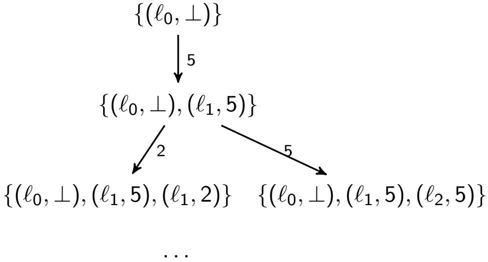

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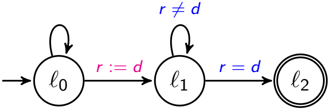

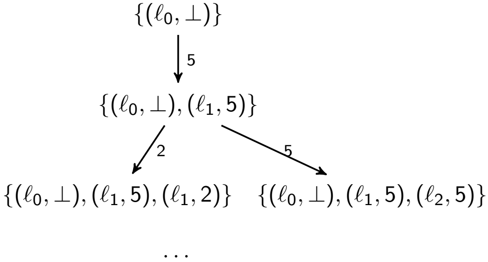

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- Idea is to prove that in an unambiguous universal automaton the size of a reachable set is even more bounded

Bounding the Size of Reachable Set

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- every config in a reachable set reaches an accepting state, and
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Let $\mathcal{A}$ have a single register, let $(\ell, 1), (\ell, 2)$ be in a reachable set.

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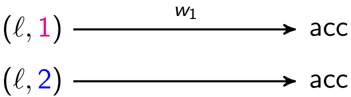

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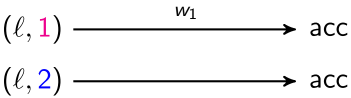

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$$(\ell, 1) \xrightarrow{w_1 = 9\,8\,2\,9\,3} \text{acc}$$

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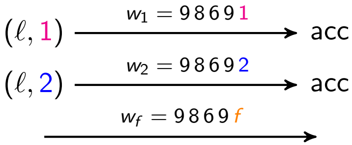

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- Similar situation for any number k of registers, e.g. for
  - $k = 2$, we cannot have $(\ell, 1, 2), (\ell, 3, 4), (\ell, 5, 6)$
  - $k = 3$, we cannot have $(\ell, 1, 2, 3), (\ell, 4, 5, 6), (\ell, 7, 8, 9), (\ell, 10, 11, 12)$
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- There is a finite constant M_k that upper bounds the number of configurations in a reachable set with the same state
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