Weighted context-free grammars over a complete strong bimonoid

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Weighted context-free grammars

Weighted context-free grammars *WCFG*

- ⋆ Quantitative features to computational processes – assigned by ***weighted models***
- ⋆ *WCFG* – classical context-free grammars in which the productions ***carry weights***
- ⋆ Weighted context-free grammars – generative model for characterisation of weighted context-free languages

Applications

- **compilers' development** (1997) – D. C. Kozen
- **model checking** (2007) – G. Hughes, T. Bultan
- **parameterized verification** (1995) – E. M. Clarke, O. Grumberg, S. Jha
- **applied to parsing and tagging tasks in the language processing** (2007) – N. A. Smith, M. Johnson

Weighted context-free grammars

Weights are assumed to form the algebraic structure of a ***semiring***, traditionally.

Semiring structure is not sufficient to describe operations needed in modern practical applications.

Computational models over more general structures

- **Shang, Lu and Lu** (2012) – studied *WCFGs* with weights taken from lattice-ordered quantum MV algebras
- **Droste and Vogler** (2014) – studied *CFGs* over unital valuation monoids
- **Rahonis and Torpari** (2019) – studied *CFGs* over bimonoids

Context-free grammars over complete strong bimonoids

Two ways for defining behaviour of such grammars:

- ⋆ **depth-first semantics** ;
- ⋆ **breath-first semantics** .

Remark

These two semantics coincide if underlying weight-structure is a semiring.

Bimonoid K

- ⋆ structure $(K, +, \cdot, 0, 1)$ such that $(K, +, 0)$ and $(K, \cdot, 1)$ are ***monoids***

Strong bimonoid K

- ⋆ $(K, +, \cdot, 0, 1)$ – bimonoid
- ⋆ $+$ is commutative operation
- ⋆ 0 acts as multiplicative zero, i.e.

$$a \cdot 0 = 0 = 0 \cdot a, \quad a \in K.$$

Algebra is **locally finite** if any of its finitely generated subalgebras is finite.

Strong bimonoid K

- ⋆ is *additively locally finite* if the monoid $(K, +, 0)$ is locally finite;
- ⋆ is *multiplicatively locally finite* if the monoid $(K, \cdot, 1)$ is locally finite;
- ⋆ is *bi-locally finite* if it is both additively and multiplicatively locally finite.

Example – bi-locally finite strong bimonoid

$(\{0\} \cup [\lambda, 1], +, \cdot, 0, 1)$

- ⋆ $\lambda < \frac{1}{2}$
- ⋆ product – $a \cdot b = \begin{cases} a \cdot b, & \text{if } a \cdot b \geqslant \lambda \\ 0, & \text{if } a \cdot b < \lambda \end{cases}$
- ⋆ addition – $a + b = \min\{a + b, 1\}$,

  It is not a locally finite strong bimonoid.

Infinitary sum operation $\sum_I$

- ⋆ I – index set
- ⋆ $\sum_I : M^I \to M$ associates with $\{m_i \mid i \in I)\} \subseteq M$ an element $\sum_{i\in I} m_i$ of M.

Commutative monoid $(M, +, 0)$ – complete

- ⋆ $(M, +, 0)$ has infinitary sum operations $\sum_I$:

$$\sum_{i\in\emptyset} m_i = 0, \; \sum_{i\in\{j\}} m_i = m_j, \; \sum_{i\in\{j,k\}} m_i = m_j + m_k, \text{ for } j \neq k,$$
$$\sum_{j\in J}\Big(\sum_{i\in I_j} m_i\Big) = \sum_{i\in I} m_i, \text{ if } \bigcup_{j\in J} I_j = I \text{ and } I_j \cap I_{j'} = \emptyset \text{ for } j \neq j'$$

Formal power series over X and K

- ⋆ mapping $\varphi : X^* \to K$
- ⋆ $K\langle\langle X^*\rangle\rangle$ – set of all series over X and K

Context-free grammar *CFG*

- a tuple $G = (N, X, \sigma, P)$ ($G = (V, \sigma, P)$) such that
  - N is a finite set of *nonterminals*
  - X is a finite set of *terminals*, such that $N \cap X = \emptyset$
  - $\sigma \in N$ is an *initial nonterminal*
  - $P \subset N \times V^*$ is a finite set of *production rules*, where $V = N \cup X$.

Basic notations

- $\alpha \to s$ – production rule $(\alpha, s) \in P$
- $\stackrel{p}{\Rightarrow}$ – *direct production* on V^* consisting of all pairs of the form $(u\alpha v, usv)$ with $u, v \in V^*$, for production rule $p = \alpha \to s$
- $\Delta = (u_0, p_1, u_1, p_2, u_2, \dots, p_n, u_n)$, $u_0, u_1, \dots, u_n \in V^*$, $p_1, \dots, p_n \in P$ such that
$$u_{i-1} \stackrel{p_i}{\Rightarrow} u_i, \quad \text{for every } i \in \{1, 2, \dots, n\}.$$
- Δ – *derivation* of u_n from u_0 in G, and we write $u_0 \stackrel{\Delta}{\Rightarrow} u_n$
- (u_{i-1}, p_i, u_i), for $i \in \{1, 2, \dots, n\}$ – *direct derivation* belonging to Δ
- $p_1 p_2 \dots p_n$ – the *production sequence* of Δ denoted by $\pi(\Delta)$
- $|\Delta|$ – *length of the derivation* Δ
- $D(u, v)$ – set of all derivations of v from u
- $S(u, v) = \{\pi(\Delta) \mid \Delta \in D(u, v)\}$
- $D_n(u, v) = \{\Delta \in D(u, v) \mid |\Delta| = n\}$, $\quad S_n(u, v) = \{\pi(\Delta) \mid \Delta \in D_n(u, v)\}$, $n \in \mathbb{N}$
- $D(w) = D(\sigma, w)$, $S(w) = S(\sigma, w)$, $D_n(w) = D_n(\sigma, w)$ and $S_n(w) = S_n(\sigma, w)$

Remark

$\Delta \mapsto \pi(\Delta)$ defines a surjective mapping from $D(u,v)$ onto $S(u,v)$, as well as from $D_n(u,v)$ onto $S_n(u,v)$, for every $n \in \mathbb{N}$.

Derivations $\Rightarrow$

- ⋆ $u \Rightarrow v$ if and only if $u \overset{p}{\Rightarrow} v$, $p \in P$
- ⋆ $u \overset{0}{\Rightarrow} v$ if and only if $u = v$
- ⋆ $u \overset{n+1}{\Rightarrow} v$ if and only if $u \overset{n}{\Rightarrow} s$ and $s \Rightarrow v$, $s \in V^*$
- ⋆ $u \overset{*}{\Rightarrow} v$ if and only if $u \overset{k}{\Rightarrow} v$, $k \in \mathbb{N}^0$

Leftmost derivations $\rightrightarrows$

- ⋆ $u \overset{p}{\rightrightarrows} v$, $p \in P$
- ⋆ $D^\ell(u,v)$ – set of all left derivations of v from u
- ⋆ $S^\ell(u,v) = \{\pi(\Delta) \mid \Delta \in D^\ell(u,v)\}$
- ⋆ $D^\ell_n(u,v)$, $S^\ell_n(u,v)$, $D^\ell(w)$, $S^\ell(w)$, $D^\ell_n(w)$, $S^\ell_n(w)$

Lenguage generated by G

- $L(G) = \{w \in X^* \mid D(w) \neq \emptyset\}$ – *language generated by* G
- Language generated by a context-free grammar – *context-free language*
- If there is $w \in L(G)$ such that $|D(w)| \geqslant 2$ – G is *ambiguous*
- L is *inherently ambiguous* if each context-free grammar G with $L(G) = L$ is ambiguous

THEOREM

Let $G = (N, X, \sigma, P)$ be a context-free grammar and $u, v \in V^+$, $u \neq v$, where $V = N \cup X$.
Then $|D_1(u, v)| \geqslant 2$ if and only if there are $\alpha, \beta \in N$, $q, r, s, t \in V^*$, with $rs \neq \varepsilon$ and $k \in \mathbb{N}^0$ such that $u = q\alpha(rs)^k r\beta t$ and v is directly derived from u applying the productions $\alpha \to \alpha rs$ and $\beta \to sr\beta$ from P.

Weighted context-free grammars over a strong bimonoid

- *WCFG* $G = (N, X, \sigma, P, \mathrm{wt})$ complete strong bimonoid K where
  - (N, X, σ, P) – context-free grammar
  - mapping $\mathrm{wt} : P \to K$ – (*weight assignment*)

Remark

We say that *WCFG* G is *unambiguous* if the underlying context-free grammar is unambiguous.

Weight of a derivation $\mathrm{wt}(\Delta) \in K$

$$\mathrm{wt}(\Delta) = \mathrm{wt}(p_1) \cdot \ldots \cdot \mathrm{wt}(p_n), \ \text{ for } \ \pi(\Delta) = p_1 \ldots p_n$$

- $\mathrm{wt}(u \overset{0}{\Rightarrow} v) = 1$, if $u = v$, and $\mathrm{wt}(u \overset{0}{\Rightarrow} v) = 0$, if $u \neq v$
- $\mathrm{wt}(u \Rightarrow v) = \sum_{p \in S_1(u,v)} \mathrm{wt}(p)$
- $\mathrm{wt}(u \overset{n+1}{\Rightarrow} v) = \sum_{s \in W_n(u)} \mathrm{wt}(u \overset{n}{\Rightarrow} s) \cdot \mathrm{wt}(s \Rightarrow v)$, where $W_n(u) = \{t \in V^* \mid u \overset{n}{\Rightarrow} t\}$

Depth-first semantics

- ★ $[\![G]\!]_{\mathrm{d}} \in K\langle\langle X^* \rangle\rangle$ – d-*behavior* of G is

$$([\![G]\!]_{\mathrm{d}}, w) = \sum_{\Delta \in D(w)} \mathrm{wt}(\Delta) = \sum_{n \in \mathbb{N}} \Big(\sum_{\Delta \in D_n(w)} \mathrm{wt}(\Delta) \Big)$$

Breath-first semantics

- ★ $[\![G]\!]_{\mathrm{b}} \in K\langle\langle X^* \rangle\rangle$ – b-*behavior* of G is

$$([\![G]\!]_{\mathrm{b}}, w) = \mathrm{wt}(\sigma \overset{*}{\Rightarrow} w) = \sum_{n \in \mathbb{N}} \mathrm{wt}(\sigma \overset{n}{\Rightarrow} w)$$

THEOREM

Let K be a complete semiring. Then $[\![G]\!]_{\mathrm{d}} = [\![G]\!]_{\mathrm{b}}$, for every weighted context-free grammar G over K.

THEOREM

Let K be a complete strong bimonoid. Then $[\![G]\!]_{\mathrm{d}} = [\![G]\!]_{\mathrm{b}}$, for every weighted context-free grammar G over K without multiple direct derivations if and only if K is right distributive.

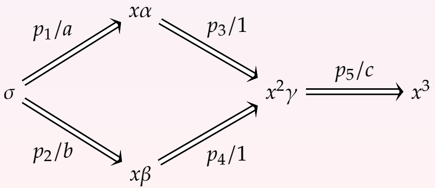
EXAMPLE

$G = (N, X, \sigma, P)$ – weighted context-free grammar with $X = \{x\}$, $N = \{\sigma, \alpha, \beta, \gamma\}$ and $a, b, c \in K$

$$p_1 = \sigma \to x\alpha, \quad p_2 = \sigma \to x\beta, \quad p_3 = \alpha \to x\gamma, \quad p_4 = \beta \to x\gamma, \quad p_5 = \gamma \to x,$$
$$\mathrm{wt}(p_1) = a, \quad \mathrm{wt}(p_2) = b, \quad \mathrm{wt}(p_3) = 1, \quad \mathrm{wt}(p_4) = 1, \quad \mathrm{wt}(p_5) = c$$

This grammar does not have multiple direct derivations, so $[\![G]\!]_{\mathrm{d}} = [\![G]\!]_{\mathrm{b}}$

$(a + b) \cdot c = [\![G]\!]_{\mathrm{b}}(x^3) = [\![G]\!]_{\mathrm{d}}(x^3) = a \cdot c + b \cdot c$



THEOREM

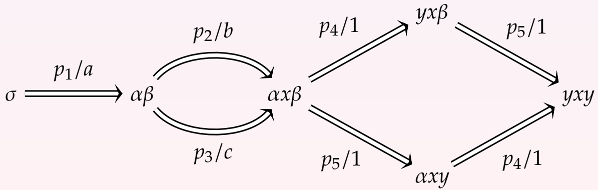
Let K be an additively idempotent complete strong bimonoid. Then $[\![G]\!]_{\mathrm{d}} = [\![G]\!]_{\mathrm{b}}$, for every weighted context-free grammar G over K if and only if K is a semiring.

EXAMPLE

$G = (N, X, \sigma, P)$ – weighted context-free grammar with $X = \{x, y\}$, $N = \{\sigma, \alpha, \beta, \gamma\}$ and $a, b, c \in K$

$$p_1 = \sigma \to \alpha\beta, \quad p_2 = \alpha \to \alpha x, \quad p_3 = \beta \to x\beta, \quad p_4 = \alpha \to y, \quad p_5 = \beta \to y,$$
$$\mathrm{wt}(p_1) = a, \quad \mathrm{wt}(p_2) = b, \quad \mathrm{wt}(p_3) = c, \quad \mathrm{wt}(p_4) = 1, \quad \mathrm{wt}(p_5) = 1.$$

$$a \cdot (b + c) = a \cdot (b + c) \cdot 1 \cdot 1 + a \cdot (b + c) \cdot 1 \cdot 1 = [\![G]\!]_{\mathrm{b}}(yxy) = [\![G]\!]_{\mathrm{d}}(yxy)$$
$$= a \cdot b \cdot 1 \cdot 1 + a \cdot b \cdot 1 \cdot 1 + a \cdot c \cdot 1 \cdot 1 + a \cdot c \cdot 1 \cdot 1 = a \cdot b + a \cdot c$$



WCFG with leftmost derivations

- $G = (N, X, \sigma, P, \mathrm{wt})$ – weighted context-free grammar
- Derivations can be constrained to leftmost derivations (no multiple leftmost direct derivations)
- *leftmost depth-first semantics* and the *leftmost breadth-first semantics* of G follows.

THEOREM

Let K be a strong bimonoid. Then $[\![G]\!]_{\ell\mathrm{d}} = [\![G]\!]_{\ell\mathrm{b}}$, for every weighted context-free grammar G over K if and only if K is right distributive.

Definition

$G = (N, X, \sigma, P, \mathrm{wt})$, $G' = (N', X', \sigma', P', \mathrm{wt}')$ – *WCFG*s over a strong bimonoid K

- G and G' are *weakly equivalent* if $L(G) = L(G')$.
- G and G' are *d strongly equivalent* if $L(G) = L(G')$ and for each $u \in X^*$ is $\mathrm{wt}(u)$ in G equal to $\mathrm{wt}'(u)$ in G' ($[\![G]\!]_{\mathrm{d}} = [\![G']\!]_{\mathrm{d}}$).

Normal forms for *WCFG*

THEOREM

Let K be a strong bimonoid and $G = (N, X, \sigma, P, \mathrm{wt})$ *WCFG* over K. There exists a *WCFG* $G' = (N', X', \sigma', P', \mathrm{wt}')$ such that G' is an extension of G with the identity projection function and $L(G) = L(G')$ and $[\![G]\!]_{\mathrm{d}} = [\![G']\!]_{\mathrm{d}}$.

THEOREM

Let $G = (N, X, \sigma, P, \mathrm{wt})$ be a *WCFG* over a strong bimonoid K. Then we can effectively construct a *WCFG* $G' = (N', X', \sigma', P', \mathrm{wt}')$ without trivial rules $p = \alpha \to \alpha$ ($\mathrm{wt}(p) = 0$) such that $L(G) = L(G')$ and $[\![G]\!]_{\mathrm{d}} = [\![G']\!]_{\mathrm{d}}$.

THEOREM

Let $G = (N, X, \sigma, P, \mathrm{wt})$ be a *WCFG* without trivial rules over a strong bimonoid K. Then we can effectively construct a *WCFG* $G' = (N', X', \sigma', P', \mathrm{wt}')$ in Chomsky normal form such that $L(G) = L(G')$ and $[\![G]\!]_{\mathrm{d}} = [\![G']\!]_{\mathrm{d}}$.

THEOREM

Let $G = (N, X, \sigma, P, \mathrm{wt})$ be a *WCFG* without trivial rules over a strong bimonoid K. Then we can effectively construct a *WCFG* $G' = (N', X', \sigma', P', \mathrm{wt}')$ in Greibach normal form such that $L(G) = L(G')$ and $[\![G]\!]_{\mathrm{d}} = [\![G']\!]_{\mathrm{d}}$.

THANK YOU FOR YOUR ATTENTION!!!