Separating words with weighted automata

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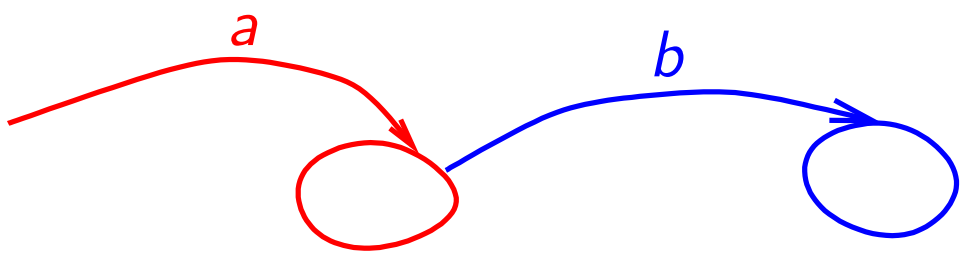

The weighted case

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- A weighted automaton $\mathcal{A}$ *separates* two words u and v if $\mathcal{A}(u) \neq \mathcal{A}(v)$.
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- Consider a and b such that $u_i = a$ and $v_i = b$ at some i.
- Set $\hat{x}_i = a_i$ is $x_i = a_i$, otherwise $\hat{x}_i = b_i$.
- $\mathcal{A}$ separates $\hat{u}$ and $\hat{v}$. Add labels to all transitions labelled b.

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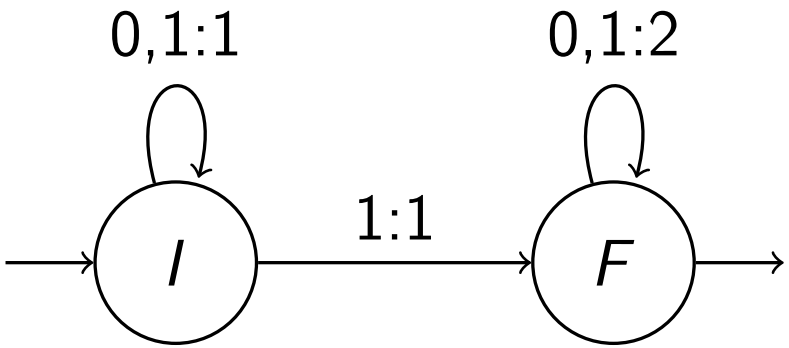
- $sep_{(+,\times)-\mathrm{WFA}}(N) = 2$:



State of the art

$\mathcal{C}$	$sep_{\mathcal{C}}(N) \leq$	$sep_{\mathcal{C}}(N) \geq$
Rev. DFA	$O(\sqrt{N})$ [Robson 96]	$3/2 \log(N) + o(\log(N))$ [Bulatov et al. 16]
DFA	$O\left(N^{2/5} \log^{3/5}(N)\right)$ [Robson 89]	$\log(N) + o(\log(N))$ [Demaine et al. 11]
NFA		$\log(N) + o(\log(N))$ [Demaine et al. 11]
WDFA		$1/2 \log(N) + o(\log(N))$
$(\max, +)$-WFA		$\Omega\left(\sqrt{\log(N)}\right)$ [Izhakian-M. 18]
$(+, \times)$-WFA	2	2

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Theorem (Izahkian-M. 2018)

$\mathcal{M}_n(\mathbb{R}_{\max})$ satisfies a nontrivial semigroup identity.
Its length grows with n as $e^{Cn^2+o(n^2)}$ for some $C \leq \frac{1+\ln(2)}{2}$.

Shitov's induction

Lemma (After Shitov 2014)

Let $W, U, V \in \mathcal{M}_n$ such that $W = PQ$, where $P \in \mathcal{M}_{n\times k}$, $Q \in \mathcal{M}_{k\times n}$, and let $s \in \{a, b\}^+$. Then

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Theorem (Shitov 2014)

$\mathcal{M}_{3\times 3}(\mathbb{R}_{\max})$ satisfies a non-trivial semigroup identity

Izhakian's theorem for regular matrices

Theorem (Izhakian 2016, Izhakian-M. 2018)

Suppose that $(q, r) \in \mathrm{Id}(\mathcal{U}_n)$, *and that* $A, B \in \mathcal{M}_n$ *satisfy*

$$\mathrm{per}(A) = \mathrm{tr}(A), \ \ \mathrm{per}(B) = \mathrm{tr}(B)$$

$$\text{and} \ \ \mathrm{rk}_{\mathrm{tr}}(q[A, B]) = \mathrm{rk}_{\mathrm{tr}}(r[A, B]) = n.$$

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$$w = (qr)[a^{\overline{n}}, b^{\overline{n}}], \ u = q[a^{\overline{n}}, b^{\overline{n}}], \ v = r[a^{\overline{n}}, b^{\overline{n}}]$$

Powers of matrices and Ranks

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For any $A \in \mathcal{M}_n(\mathbb{R}_{\max})$ *and* $t \geq (n-1)^2 + 1$, *we have*

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Theorem (After M.-Nowak-Sergeev 2014)

For $t \geq (n-1)^2 + 1$, *there is decomposition indexed by a collection of node-disjoint elementary cycles :*

$$A^t = \bigvee_{\theta \in \Theta} C_\theta (S_\theta)^t R_\theta,$$

the sum of the lengths of the cycles being at most $\mathrm{rk}_f(A)$.

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$$(ua, va)\left[((qr)^t w_1\,[q, r])\,[a^{\overline{n}}, b^{\overline{n}}], ((qr)^t w_2\,[q, r])\,[a^{\overline{n}}, b^{\overline{n}}]\right] \in \mathrm{Id}(\mathcal{M}_n).$$

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$$\left.\begin{array}{rcc} \overline{n} & = & e^{n+o(n)} \\ |q| = |r| & = & 2^{n+o(n)} \\ t & = & O(n^2) = e^{o(n)} \\ |w_i| & \leq & 1 \end{array}\right\} \Rightarrow |u_{n+1}| = e^{(1+\ln 2)n}|u_n|.$$

Thank you for your attention.