Decidability of crisp-determinization for weighted finite automata over past-finite monotonic strong bimonoids

M. Droste Z. Fülöp D. Kószó[1] H. Vogler

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Weighted finite automata

Σ: alphabet

strong bimonoid: $(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ [Ćirić, Droste, Ignjatović, Vogler 10]
semiring without distributivity required [Droste, Stüber, Vogler 10]

(Σ, B)-wfa: $\mathcal{A} = (Q, I, \delta, F)$ with $\delta : Q \times \Sigma \times Q \to B$

$\mathcal{A}$ computes the weighted language $[\![\mathcal{A}]\!] : \Sigma^* \to B$

for each $w \in \Sigma^*$:

$$[\![\mathcal{A}]\!](w) = \bigoplus_{p,q \in Q} \bigoplus_{\rho \in \mathrm{R}_{\mathcal{A}}(p,w,q)} I_p \otimes \mathbf{wt}(w, \rho) \otimes F_q$$

image of $[\![\mathcal{A}]\!]$: $\mathrm{im}([\![\mathcal{A}]\!]) = \{[\![\mathcal{A}]\!](w) \mid w \in \Sigma^*\}$

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$\mathcal{A} = (Q, I, \delta, F)$: (Σ, B)-wfa

$\mathcal{A}$ is

- unambiguous: for each $w \in \Sigma^*$ there is at most one $\rho \in \mathrm{R}_{\mathcal{A}}(p, w, q)$ for some $p, q \in Q$ such that $I_p \neq \mathbb{0}$ and $F_q \neq \mathbb{0}$
- trim: each $r \in Q$ is useful
- crisp-deterministic: for every $p \in Q$ and $\sigma \in \Sigma$ there is a $q \in Q$:
  - $\delta(p, \sigma, q) = \mathbb{1}$ and
  - $\delta(p, \sigma, q') = \mathbb{0}$ for each $q' \in Q \setminus \{q\}$

crisp-determinization: to find a crisp-deterministic (Σ, B)-wfa $\mathcal{B}$ such that $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$ if possible

Crisp-determinization

Σ: alphabet
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Theorem. [Fülöp, Kószó, Vogler 19]

It is undecidable whether, for arbitrary deterministic (Σ, B)-wfa $\mathcal{A}$ given effectively, there exists a crisp-deterministic (Σ, B)-wfa $\mathcal{B}$ such that $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$.

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Lemma. [Droste, Stüber, Vogler 10]

Let $\mathcal{A}$ be a (Σ, B)-wfa. The following statements are equivalent.

1. There exists a crisp-deterministic (Σ, B)-wfa $\mathcal{B}$ such that $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$.
2. $\mathrm{im}([\![\mathcal{A}]\!])$ is finite and $[\![\mathcal{A}]\!]^{-1}(b) \subseteq \Sigma^*$ is recognizable for each $b \in \mathrm{im}([\![\mathcal{A}]\!])$.

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Past-finite monotonic strong bimonoids

strong bimonoid $(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ is monotonic: [Borchardt, Fülöp, Gazdag, Maletti 05]
if there is a partial order $\preceq$ on B such that

(i) for every $a, b \in B$: $a \preceq a \oplus b$

(ii) for every $a, b, c \in B \setminus \{\mathbb{0}\}$ with $b \neq \mathbb{1}$: $a \otimes c \prec a \otimes b \otimes c$

strong bimonoid B is past-finite monotonic: [Droste, Fülöp, Kószó, Vogler 20]

(i) B is monotonic

(ii) $\mathbf{past}(b) = \{a \in B \mid a \preceq b\}$ is finite for each $b \in B$

Past-finite monotonic strong bimonoids

Examples:

- semiring of natural numbers $(\mathbb{N}, +, \cdot, 0, 1, \leq)$
- arctic semiring $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0, \leq)$
- $(\mathbb{N}, \mathrm{lcm}, \cdot, 0, 1, \leq)$ lcm = least common multiple
- given: $(B, \preceq)$ past-finite poset, $(B, +)$ monotonic comm. semigroup, $(B, \times)$ monotonic semigroup

  construct: past-finite monotonic strong bimonoid $(B \cup \{\mathbb{0}, \mathbb{1}\}, \oplus, \otimes, \mathbb{0}, \mathbb{1}, \preceq')$

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$\mathcal{A} = (Q, I, \delta, F)$: (Σ, B)-wfa

short string: a string $u \in \Sigma^*$ with $|u| < |Q|$

short loop: a run $\kappa \in \mathrm{R}_{\mathcal{A}}(r, u, r)$ for some $r \in Q$ and short string $u \in \Sigma^*$

short loops of $\mathcal{A}$ have weight $\mathbb{1}$: each short loop has weight $\mathbb{1}$

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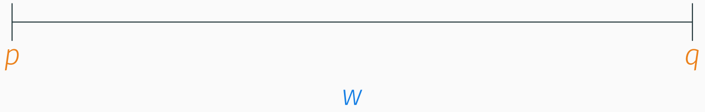

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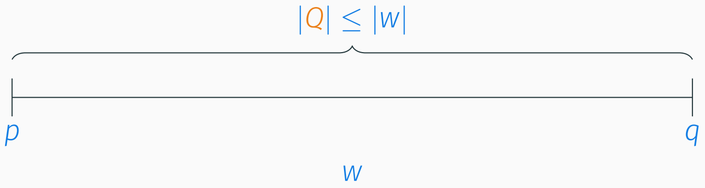

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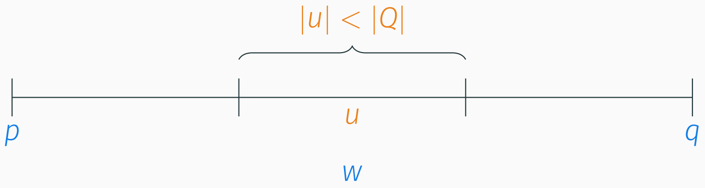

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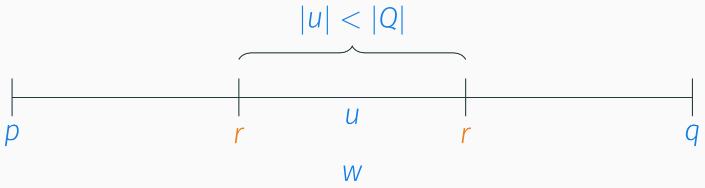

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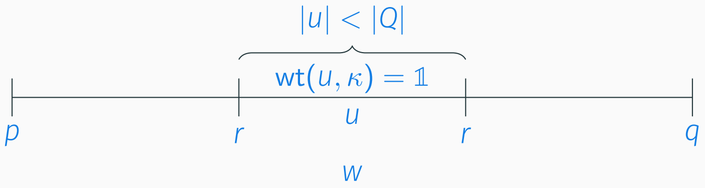

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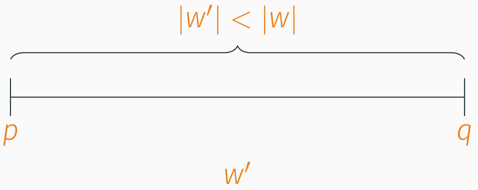

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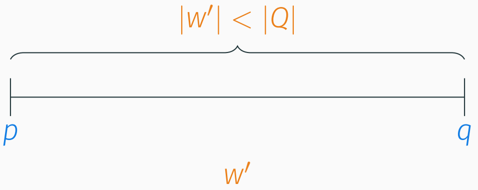

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Lemma. [Droste, Fülöp, Kószó, Vogler 20]

Let $\mathcal{A}$ be trim. If (a) short loops of $\mathcal{A}$ have weight $\mathbb{1}$ and (b) B is additively locally finite or $\mathcal{A}$ is unambiguous, then $\mathrm{im}(\llbracket\mathcal{A}\rrbracket)$ is finite.

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FINITE!

$\Downarrow$

$\mathrm{C}(\mathcal{A})$ set of all complete run weights: $I_p \otimes \mathrm{wt}(w, \rho) \otimes F_q$

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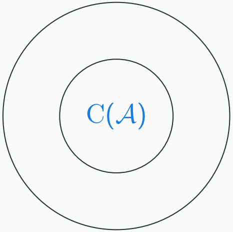

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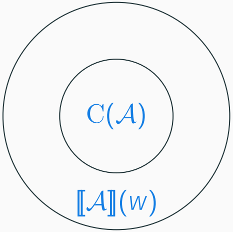

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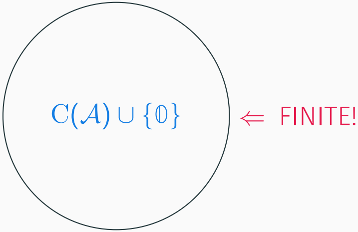
Let $\mathcal{A}$ be trim. If (a) short loops of $\mathcal{A}$ have weight $\mathbb{1}$ and (b) B is additively locally finite or $\mathcal{A}$ is unambiguous, then $\mathrm{im}(\llbracket \mathcal{A} \rrbracket)$ is finite.

Proof.



Decidability of finiteness of $\mathrm{im}(\llbracket\mathcal{A}\rrbracket)$

Σ: alphabet
$(B, \oplus, \otimes, \mathbb{0}, \mathbb{1}, \preceq)$: monotonic s.b.
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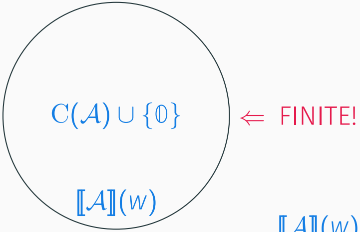
Proof. Let $w \in \Sigma^*$.



$\llbracket\mathcal{A}\rrbracket(w) = \mathbb{0}$ or $\llbracket\mathcal{A}\rrbracket(w) = I_p \otimes \mathrm{wt}(w, \rho) \otimes F_q$

□

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Let B be past-finite and $\mathcal{A}$ be trim.

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Proof. by contraposition

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Proof. by contraposition

construct: $u_1, u_2, u_3, \ldots$ with $[\![\mathcal{A}]\!](u_1), [\![\mathcal{A}]\!](u_2), [\![\mathcal{A}]\!](u_3), \ldots$ pairwise different

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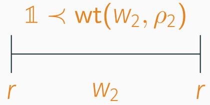

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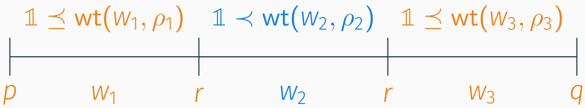

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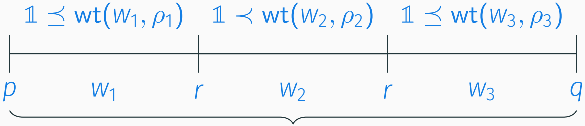

$u_1 = w_1 w_2 w_3$ and $\mathrm{wt}(u_1, \rho_1\rho_2\rho_3) \preceq \llbracket \mathcal{A} \rrbracket(u_1)$

construct: $u_1, u_2, u_3, \ldots$ with $\llbracket \mathcal{A} \rrbracket(u_1), \llbracket \mathcal{A} \rrbracket(u_2), \llbracket \mathcal{A} \rrbracket(u_3), \ldots$ pairwise different

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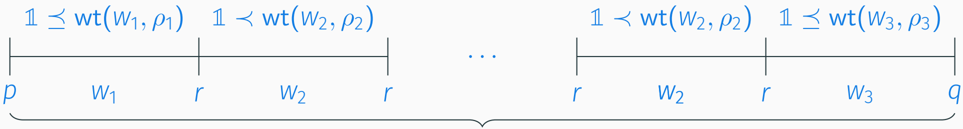

$u_2 = w_1 w_2^{n_2} w_3$, $\mathrm{wt}(u_2, \rho_1 \rho_2^{n_2} \rho_3) \notin \mathrm{past}(\llbracket \mathcal{A} \rrbracket(u_1))$, and $\mathrm{wt}(u_2, \rho_1 \rho_2^{n_2} \rho_3) \preceq \llbracket \mathcal{A} \rrbracket(u_2)$

construct: $u_1, u_2, u_3, \dots$ with $\llbracket \mathcal{A} \rrbracket(u_1), \llbracket \mathcal{A} \rrbracket(u_2), \llbracket \mathcal{A} \rrbracket(u_3), \dots$ pairwise different

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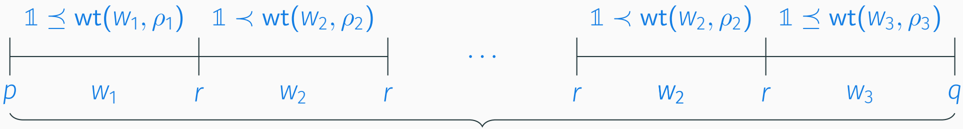

$u_3 = w_1 w_2^{n_3} w_3$, $\mathrm{wt}(u_3, \rho_1 \rho_2^{n_3} \rho_3) \notin \mathrm{past}(\llbracket\mathcal{A}\rrbracket(u_1)) \cup \mathrm{past}(\llbracket\mathcal{A}\rrbracket(u_2))$, and $\mathrm{wt}(u_3, \rho_1 \rho_2^{n_3} \rho_3) \preceq \llbracket\mathcal{A}\rrbracket(u_3)$

construct: $u_1, u_2, u_3, \dots$ with $\llbracket\mathcal{A}\rrbracket(u_1), \llbracket\mathcal{A}\rrbracket(u_2), \llbracket\mathcal{A}\rrbracket(u_3), \dots$ pairwise different □

Decidability of finiteness of $\mathrm{im}(\llbracket\mathcal{A}\rrbracket)$

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Lemma. [Droste, Fülöp, Kószó, Vogler 20]

Let B be past-finite and $\mathcal{A}$ be trim. If B is additively locally finite or $\mathcal{A}$ is unambiguous, then the following statements are equivalent.

1. $\mathrm{im}(\llbracket\mathcal{A}\rrbracket)$ is finite.
2. Short loops of $\mathcal{A}$ have weight $\mathbb{1}$.

Decidability of finiteness of $\mathrm{im}(\llbracket\mathcal{A}\rrbracket)$

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1. $\mathrm{im}(\llbracket\mathcal{A}\rrbracket)$ is finite.
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Corollary. [Droste, Fülöp, Kószó, Vogler 20]

Let B be past-finite and $\mathcal{A}$ be trim. If B is additively locally finite or $\mathcal{A}$ is unambiguous, then it is decidable whether $\mathrm{im}([\![\mathcal{A}]\!])$ is finite.

Decidability of crisp-determinization

Σ: alphabet
B: monotonic s.b.
$\mathcal{A}$: (Σ, B)-wfa

Theorem. [Droste, Fülöp, Kószó, Vogler 20]

Let B be past-finite.

1. If B is additively locally finite, then it is decidable whether an arbitrary $\mathcal{A}$ given effectively is crisp-determinizable.
2. It is decidable whether an arbitrary unambiguous $\mathcal{A}$ given effectively is crisp-determinizable.

Theorem. [Droste, Fülöp, Kószó, Vogler 20b]

Let B be past-finite and $\mathcal{A}$ be given effectively. If $\mathcal{A}$ is crisp-determinizable, then we can construct a crisp-deterministic (Σ, B)-wfa $\mathcal{B}$ such that $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$.

Decidability of crisp-determinization

Σ: alphabet
B: monotonic s.b.
$\mathcal{A}$: (Σ, B)-wfa

Theorem. [Droste, Fülöp, Kószó, Vogler 20b]

Let B be past-finite. It is decidable, for every $\mathcal{A}$ given effectively and $k \in \mathbb{N}$, whether we have $|\mathrm{im}([\![\mathcal{A}]\!])| \leq k$. Moreover, in this case we can construct a crisp-deterministic (Σ, B)-wfa $\mathcal{B}$ such that $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$.

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Strong bimonoids

More examples [Droste, Stüber, Vogler 10]

- $(C, +, 0)$: commutative monoid
  then $(B, \oplus, \circ, \widetilde{0}, \mathrm{id})$ strong bimonoid
  - B: set of all mappings $f : C \to C$ such that $f(0) = 0$
  - $\oplus$: pointwise addition
  - $\circ$: composition of mappings
  - $\widetilde{0}$: constant mapping zero
  - id: identity mapping
  - satisfying only one distributivity law
- $(\Sigma^* \cup \{\infty\}, \mathrm{lcp}, \cdot, \infty, \varepsilon)$
  - lcp = longest common prefix
  - for each $s \in \Sigma^* \cup \{\infty\}$:
    $\mathrm{lcp}(s, \infty) = s = \mathrm{lcp}(\infty, s)$ and $s \cdot \infty = \infty = \infty \cdot s$
  - only left distributive

Past-finite monotonic strong bimonoids

More examples:

[Droste, Vogler 12]

[Droste, Fülöp, Kószó, Vogler 20]

$(\mathbb{N}, +', \cdot, 0, 1, \leq)$ with $+'$ defined, $\forall a, b \in \mathbb{N}$, by

$$a +' b = \begin{cases} \min\{a + b, 100\} & \text{if } a, b \leq 100 \\ \max\{a, b\} & \text{otherwise} \end{cases}$$

Past-finite monotonic strong bimonoids

More examples:

[Droste, Vogler 12]

[Droste, Fülöp, Kószó, Vogler 20]

$(B, \prec)$: past-finite partially ordered set
$(B, +)$: commutative semigroup and $\forall a, b \in B$: $a \preceq a + b$
$(B, \times)$: semigroup and $\forall a, b, c \in B$: $a \prec a \times b$, $c \prec b \times c$, and $a \times c \prec a \times b \times c$
then construct $(B', \oplus, \otimes, \mathbb{0}, \mathbb{1}, \prec')$

- $B' = B \cup \{\mathbb{0}, \mathbb{1}\}$ such that $\mathbb{0}, \mathbb{1} \notin B$
- $\oplus|_{B \times B} = +$ and $\forall b \in B$ we let $\mathbb{0} \oplus b = b = b \oplus \mathbb{0}$, and if $b \neq \mathbb{0}$, then $\mathbb{1} \oplus b = b = b \oplus \mathbb{1}$
- $\otimes|_{B \times B} = \times$ and $\forall b \in B$ we let $\mathbb{0} \otimes b = \mathbb{0} = b \otimes \mathbb{0}$, and $\mathbb{1} \otimes b = b = b \otimes \mathbb{1}$
- $\mathbb{0} \prec' \mathbb{1} \prec' b$ for each $b \in B$ and $\preceq' \cap (B \times B) = \preceq$

Crisp-determinization

Σ: alphabet
$(B, \oplus, \otimes, \mathbb{0}, \mathbb{1})$: strong bimonoid
$\mathcal{A} = (Q, I, \delta, F)$: (Σ, B)-wfa
$I : Q \to B$, $\delta : Q \times \Sigma \times Q \to B$, $F : Q \to B$

run weights of $\mathcal{A}$:

$$\mathrm{H}(\mathcal{A}) = \{\mathrm{wt}(w, \rho) \mid p, q \in Q, w \in \Sigma^*, \text{ and } \rho \in \mathrm{R}_{\mathcal{A}}(w)\}$$

complete run weights of $\mathcal{A}$:

$$\mathrm{C}(\mathcal{A}) = \{I_p \otimes \mathrm{wt}(w, \rho) \otimes F_q \mid p, q \in Q, w \in \Sigma^*, \text{ and } \rho \in \mathrm{R}_{\mathcal{A}}(p, w, q)\}$$

complete run number mapping of b in $\mathrm{C}(\mathcal{A})$: $f_{\mathcal{A},b} : \Sigma^* \to \mathbb{N}$

$$f_{\mathcal{A},b}(w) = |\{\rho \in \mathrm{R}_{\mathcal{A}}(p, w, q) \mid p, q \in Q \text{ and } I_p \otimes \mathrm{wt}(w, \rho) \otimes F_q = b\}|$$

Crisp-determinization

Σ: alphabet
B: strong bimonoid

Theorem. [Droste, Fülöp, Kószó, Vogler 20b]

Let $\mathcal{A}$ be a (Σ, B)-wfa such that $\mathrm{H}(\mathcal{A})$ is finite. If, for each $b \in \mathrm{C}(\mathcal{A})$, the mapping $f_{\mathcal{A},b}$ is bounded or b has finite additive order, then the following statements hold.

1. There exists a crisp-deterministic (Σ, B)-wfa $\mathcal{B}$ such that $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$.
2. If $\mathcal{A}$ is given effectively, then we can construct this $\mathcal{B}$.

Crisp-determinization

Σ: alphabet
B: monotonic s.b.

Theorem. [Droste, Fülöp, Kószó, Vogler 20b]

Let B be past-finite and $\mathcal{A}$ be a trim (Σ, B)-wfa. Then the following statements are equivalent.

1. $\mathcal{A}$ is crisp-determinizable.
2. Short loops of $\mathcal{A}$ have weight $\mathbb{1}$ and, for each $b \in \mathrm{C}(\mathcal{A})$, the mapping $f_{\mathcal{A},b}$ is bounded or b has finite additive order.