

Posets with Interfaces

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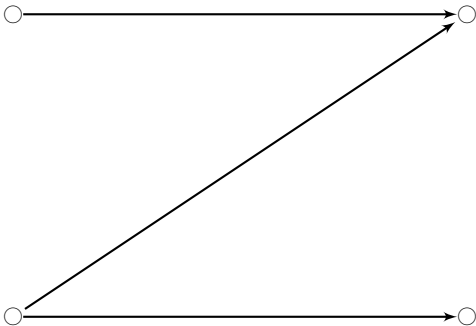
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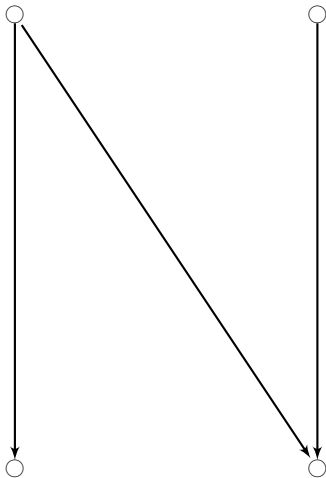
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WATA 2021

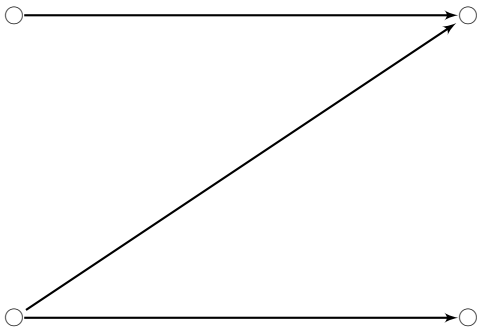
Motivation



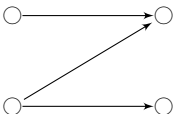
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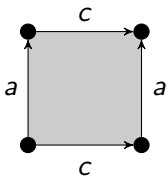


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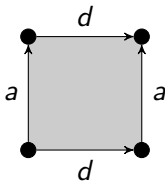


- **Kleene algebra** is nice and useful
 - ▶ also its extensions: semimodules, tests, domain, ...
- **Concurrent Kleene algebra**: extension of KA for concurrency
 - ▶ [Hoare, Möller, O'Hearn, Struth, van Staden, Villard, Wehrman, Zhu '09, '11, '16]
- Kleene algebra plus **parallel composition**
- the **free CKA** (minus some details): sets of **series-parallel pomsets**
 - ▶ labeled posets with concatenation & parallel composition
- *Something's amiss in concurrent Kleene algebra*

Example, Using Higher-Dimensional Automata

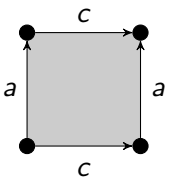


$$\begin{pmatrix} a \\ c \end{pmatrix}$$



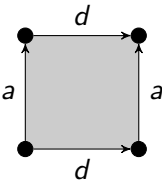
$$\begin{pmatrix} a \\ d \end{pmatrix}$$

Example, Using Higher-Dimensional Automata



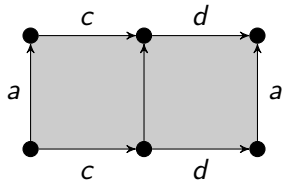
$$\begin{pmatrix} a \\ c \end{pmatrix}$$

$a *$



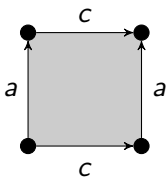
$$\begin{pmatrix} a \\ d \end{pmatrix}$$

$=$

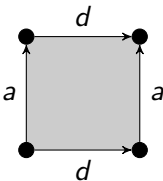


$$\begin{pmatrix} c \xrightarrow{a} d \end{pmatrix}$$

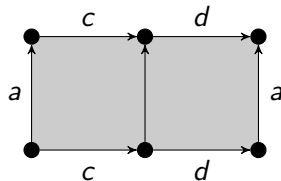
Example, Using Higher-Dimensional Automata



$$\begin{pmatrix} a \\ c \end{pmatrix}$$

 $a *$


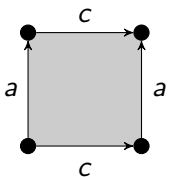
$$\begin{pmatrix} a \\ d \end{pmatrix}$$

 $=$


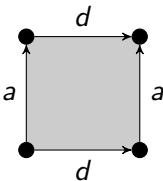
$$\begin{pmatrix} a \\ c \longrightarrow d \end{pmatrix}$$

$$\begin{pmatrix} a \\ c \end{pmatrix} \parallel \begin{pmatrix} a \\ d \end{pmatrix} = \begin{pmatrix} a \\ c \\ a \\ d \end{pmatrix} ??$$

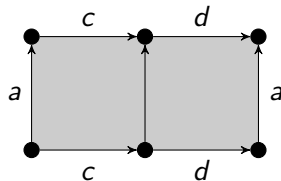
Example, Using Higher-Dimensional Automata



$$\begin{pmatrix} a \\ c \end{pmatrix}$$

 $a *$


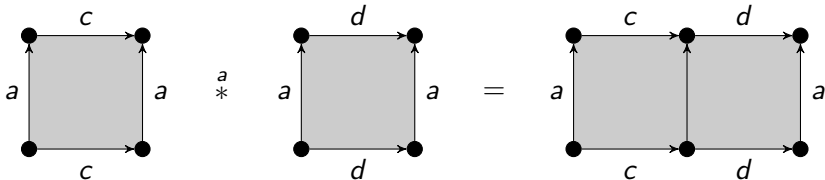
$$\begin{pmatrix} a \\ d \end{pmatrix}$$

 $=$


$$\begin{pmatrix} a \\ c \longrightarrow d \end{pmatrix}$$

$$\begin{pmatrix} a \\ c \end{pmatrix} * \begin{pmatrix} a \\ d \end{pmatrix} = \begin{pmatrix} a \xrightarrow{\quad} a \\ c \xrightarrow{\quad} d \end{pmatrix} ??$$

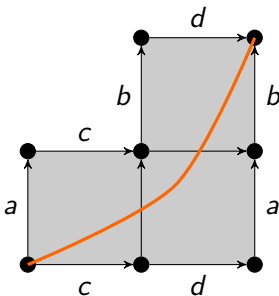
Example, Using Higher-Dimensional Automata



$$\begin{pmatrix} a \\ c \end{pmatrix} \quad \mathbf{a} * \quad \begin{pmatrix} a \\ d \end{pmatrix} \quad = \quad \begin{pmatrix} a \\ c \longrightarrow d \end{pmatrix}$$

- new **gluing** operation on pomsets, to *continue events across compositions*

Another Example



$$\begin{pmatrix} a \\ c \end{pmatrix} \overset{a}{*} \begin{pmatrix} a \\ d \end{pmatrix} \overset{d}{*} \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a \longrightarrow b \\ c \longrightarrow d \end{pmatrix}$$

- this is the **N** pomset, which is **not series-parallel**

- 1 Introduction
- 2 Series-Parallel Posets
- 3 Posets with Interfaces
- 4 Gluing-Parallel Iposets
- 5 Conclusion

Series-Parallel Posets

- a **poset**: *finite* set P plus partial order \leq : reflexive, transitive, antisymmetric
- **parallel** composition of posets $(P_1, \leq_1), (P_2, \leq_2)$:

$$P_1 \otimes P_2 = (P_1 \sqcup P_2, \leq_1 \cup \leq_2)$$

↑↑ disjoint union

- **serial** composition:

$$P_1 * P_2 = (P_1 \sqcup P_2, \leq_1 \cup \leq_2 \cup P_1 \times P_2)$$

↑↑ P_1 before P_2

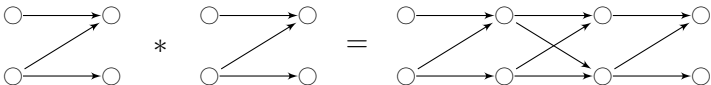
Series-Parallel Posets

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- **serial** composition:

$$P_1 * P_2 = (P_1 \sqcup P_2, \leq_1 \cup \leq_2 \cup P_1 \times P_2)$$



Series-Parallel Posets

Definition (Winkowski '77, Grabowski '81)

A poset is **series-parallel (sp)** if it is empty or can be obtained from the singleton poset by a finite number of serial and parallel compositions.

Theorem (Grabowski '81)

A poset is sp iff it does not contain \mathbf{N} as an induced subposet.

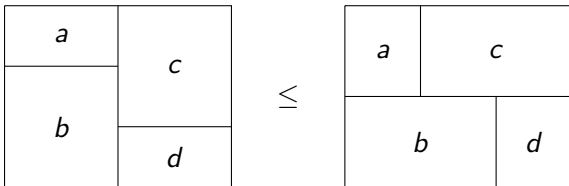
The equational theory of sp-posets is well-understood: [Gischer '88], [Bloom-Esik '96]

Concurrent Monoids

Definition (Gischer '88, Hoare *et al.* '11)

A **concurrent monoid** is an ordered bimonoid $(S, \leq, *, \parallel, 1)$ with shared $*-\parallel$ -unit 1 which satisfies **weak interchange**:

$$(a \parallel b) * (c \parallel d) \leq (a * c) \parallel (b * d)$$



- **subsumption** on posets: $P \preceq Q$ if P “has more order” than Q

Theorem (Gischer '88, Bloom-Esik '96)

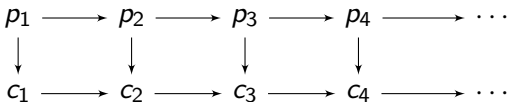
The set of sp-posets under subsumption is the free concurrent monoid.

Background: Concurrent Kleene Algebra

- concurrent monoids: basic algebraic structure for **concurrent Kleene algebra**
- Kleene algebra is useful in language theory (compilers!), verification, etc. etc.
- **concurrent** Kleene algebra: quest to extend that success to **parallel** programming
- distributed systems; weak memory; etc. etc.
- Tony Hoare, Bernhard Möller, Peter O'Hearn, Georg Struth, Huibiao Zhu 2009++
- process algebra with $+$ (non-determinism), \cdot (concatenation), \parallel (parallelism), $*$ (iteration), and \dagger (parallel iteration)
- (in this talk, no iterations!)
- **problem: no Ns!**

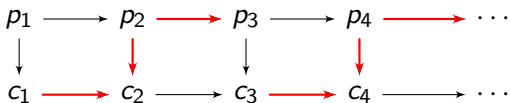
Problem & Solution

- we like the **N** poset, but it's not series-parallel
- in fact, **N**'s are everywhere: for example, *producer-consumer*:



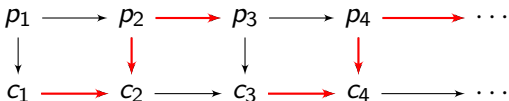
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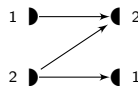
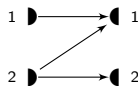
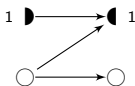
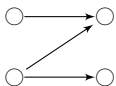
Problem

*Find a class of posets which includes **N** (and sp-posets) and which has good algebraic properties.*

Our Proposal

Posets with **interfaces** with parallel and **gluing** composition.

Posets with Interfaces



Definition

A **poset with interfaces (iposet)** is a poset P plus two injections

$$[n] \xrightarrow{s} P \xleftarrow{t} [m]$$

such that $s[n]$ is minimal and $t[m]$ is maximal in P .

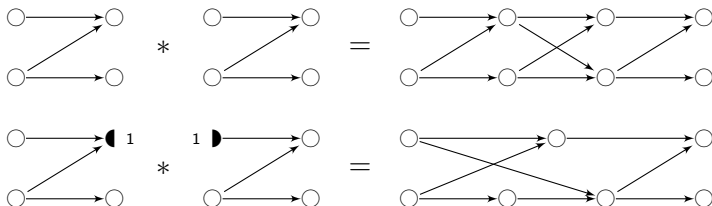
- ($[n] = \{1, \dots, n\}$; $S \subseteq P$ *minimal* if $p \not\prec s$ for all $p \in P, s \in S$)
- (there are 25 non-isomorphic iposets with underlying \mathbf{N})
- s : **starting interface** ; t : **terminating interface**
- events in $t[m]$ are *unfinished* ; events in $s[n]$ are “*unstarted*”

Gluing Composition

Definition

The **gluing composition** of iposets $s_1 : [n] \rightarrow (P_1, \leq_1) \leftarrow [m] : t_1$ and $s_2 : [m] \rightarrow (P_2, \leq_2) \leftarrow [k] : t_2$:

$$P_1 * P_2 = \begin{cases} (P_1 \sqcup P_2) / t_1(i) = s_2(i) \\ \leq_1 \cup \leq_2 \cup (P_1 \setminus t_1[m]) \times (P_2 \setminus s_2[m]) \end{cases}$$

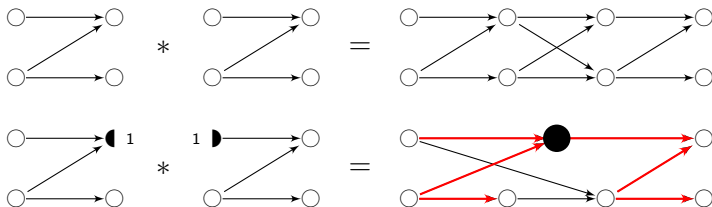


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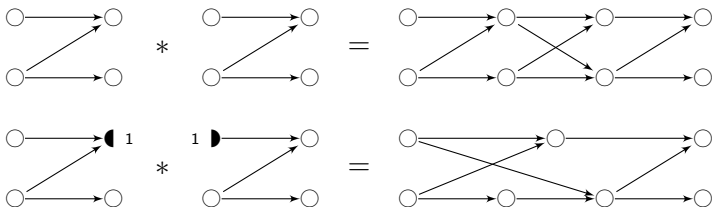


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- only defined if terminating int. of P_1 is equal to starting int. of P_2
- iposets form category (with gluing as composition)

Gluing-Parallel Iposets

- recall *sp-posets*: freely generated from \circ using $*$ and \otimes
- the four singleton iposets:



- gp-iposets**: generated from \circ , \bullet , \blacktriangleleft , \blacktriangleright using $*$ and \otimes

Theorem

Gp-iposets are **freely generated**, except for the relations

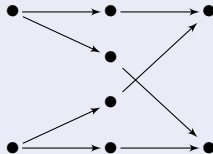
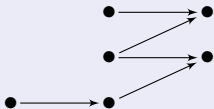
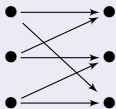
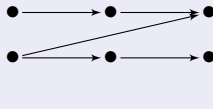
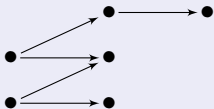
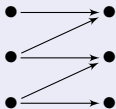
$$\begin{aligned} \left(\begin{array}{c} \blacktriangleleft 1 \\ P \end{array} \right) * \left(\begin{array}{c} 1 \blacktriangleright \\ Q \end{array} \right) &= \left(\begin{array}{c} \circ \\ P * Q \end{array} \right) & \left(\begin{array}{c} \blacktriangleleft 1 \\ P \end{array} \right) * \left(\begin{array}{c} 1 \blacktriangleright 1 \\ Q \end{array} \right) &= \left(\begin{array}{c} \blacktriangleleft 1 \\ P * Q \end{array} \right) \\ \left(\begin{array}{c} 1 \blacktriangleright 1 \\ P \end{array} \right) * \left(\begin{array}{c} 1 \blacktriangleright \\ Q \end{array} \right) &= \left(\begin{array}{c} 1 \blacktriangleright \\ P * Q \end{array} \right) & \left(\begin{array}{c} 1 \blacktriangleright 1 \\ P \end{array} \right) * \left(\begin{array}{c} 1 \blacktriangleright 1 \\ Q \end{array} \right) &= \left(\begin{array}{c} 1 \blacktriangleright 1 \\ P * Q \end{array} \right) \end{aligned}$$

Forbidden Substructures

- recall: P is sp iff P is **N**-free

Proposition

If P is gp, then it does *not* have any of the below induced subposets:



- unlike for sp -posets, that's **not an iff** (we don't know)
- but these are the only forbidden substructures for gp -posets on ≤ 9 points

Conclusion

- **posets with interfaces** for concurrency
- instead of concurrent monoid, **small category with lax tensor**
 - ▶ a “multi-object concurrent monoid”
- **gluing-parallel** iposets include sp-posets and interval orders
- generation is “almost free”
- characterization by **forbidden substructures**?

Ongoing and Future Work

- Concurrent Kleene algebra:
 - ▶ concurrent monoid \leadsto concurrent Kleene algebra
 - ▶ concurrent category \leadsto bicategory with lax tensors?
- CKA with domain:
 - ▶ domain elements are “structure-less” iposets
 - ▶ relation to higher-dimensional modal logic?
 - ▶ higher-dimensional modal Kleene algebra
- Languages of higher-dimensional automata:
 - ▶ sets of interval orders
 - ▶ concatenation of HDA \approx gluing of (sets of) interval orders
 - ▶ theory of regular languages for concurrency?
- Real-time concurrency:
 - ▶ higher-dimensional timed automata
 - ▶ languages are sets of real-time interval orders?
 - ▶ relation to real-time Petri nets?

6 Parallel Composition of Iposets

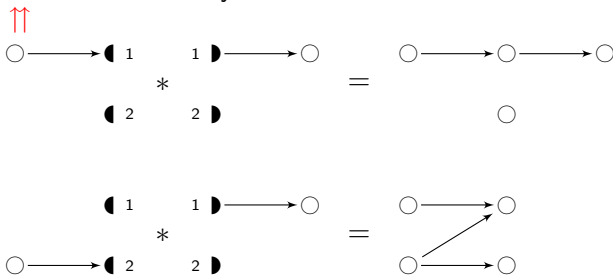
7 Interval Orders

Parallel Composition

- **parallel composition** of iposets: put posets in parallel and renumber interfaces
- for $[n_1] \rightarrow P_1 \leftarrow [m_1]$ and $[n_2] \rightarrow P_2 \leftarrow [m_2]$, have $[n_1 + n_2] \rightarrow P_1 \otimes P_2 \leftarrow [m_1 + m_2]$
- **not commutative** ; only “lax tensor” ; **not a PROP**

Parallel Composition

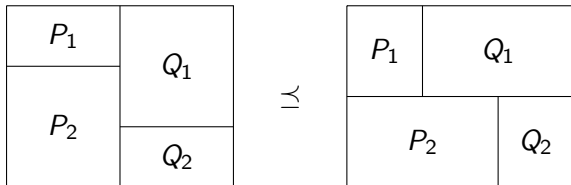
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- **not commutative** ; only “**lax tensor**” ; **not a PROP**
 \Uparrow

$$(P_1 \otimes P_2) * (Q_1 \otimes Q_2) \not\cong (P_1 * Q_1) \otimes (P_2 * Q_2)$$



6 Parallel Composition of Iposets

7 Interval Orders

Interval Orders

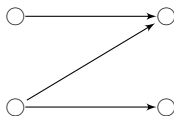
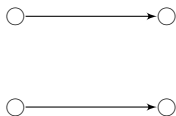


- posets which are good for concurrency?
- already in [Wiener 1914], then [Winkowski '77], [Lamport '86], [van Glabbeek '90], [Vogler '91], [Janicky '93], etc.
- **interval orders**: posets which have representation as (real) intervals, ordered by $\max_1 \leq \min_2$
- Lemma (Fishburn '70): A poset is interval iff it does not contain $\mathbb{II} = \left(\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \cdot & \longrightarrow & \cdot \end{array} \right)$ as induced subposet.
- intuitively: if $a \longrightarrow b$ and $c \longrightarrow d$, then also $a \longrightarrow d$ or $c \longrightarrow b$

Gluing of Interval Orders

$$\begin{array}{c}
 \left(\begin{array}{c} a \\ c \end{array} \right) \begin{array}{c} a \\ * \end{array} \left(\begin{array}{c} a \\ d \end{array} \right) \begin{array}{c} d \\ * \end{array} \left(\begin{array}{c} b \\ d \end{array} \right) \\
 \\
 \frac{a}{c} \text{ --- } \frac{a}{d} \text{ --- } \frac{b}{d} \\
 \\
 \frac{a}{c} \quad \frac{b}{d}
 \end{array}
 =
 \begin{array}{c}
 \left(\begin{array}{cc} a & \longrightarrow & b \\ c & \longrightarrow & d \end{array} \right) \\
 \\
 \frac{a}{c} \quad \frac{b}{d}
 \end{array}$$

Interval Orders vs Series-Parallel Posets



- **interval orders** are used in Petri net theory and distributed computing
- but have no algebraic representation (so far)
- **sp-posets** are used in concurrency theory & have nice algebraic theory
- but applicativity is doubtful!
- int. orders are **II**-free; sp-posets are **N**-free
- incomparable: **II** is sp; **N** is interval
- goal: marriage