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Greibach Normal Form and Simple Automata for Weighted ω-Context-Free Languages

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Leipzig University

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QuantLA

A Word on Weight Structures

Complete and Continuous Star-Omega Semirings

Complete: "has infinite sums and infinite products"
Continuous: existence of certain fixpoints
Additional: star * and omega $^\omega$ operation

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Examples

$\langle \mathbb{R}_+^\infty, \min, +, \infty, 0 \rangle$, $\langle \mathbb{Q}_+^\infty, +, \cdot, 0, 1 \rangle$

Greibach Normal Form for ω-Algebraic Systems

ω-Algebraic Systems

Intuition: ω-context-free grammars written as equation systems

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Example (ω-context-free grammar)

$$Y_1 \to Y_2 Y_1$$
$$Y_2 \to aY_2b \mid ab$$

(with Büchi-accepting set $\{Y_1\}$) recognizes $\{a^n b^n \mid n \geq 1\}^\omega$.

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Example (ω-algebraic system)

$$\begin{aligned} y_1 &= y_2 y_1 \\ y_2 &= a y_2 b \oplus ab \end{aligned}$$

has (the *first*, i.e., only y_1 is Büchi-accepting) *canonical solution* $\left(\bigoplus_{n\geq 1} a^n b^n\right)^\omega$.

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Algebraic Series

The canonical solutions are called *ω-algebraic series*.

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Theorem

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Let u be an ω-series over A. The following are equivalent:

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Proof Idea

Use ω-Kleene Closure.
Lift from algebraic systems of finite words.

From ω-Algebraic Systems to Automata

Transition Matrix - Büchi Condition

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Intuition: *adjacency matrix* of a finite automaton

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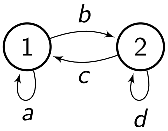

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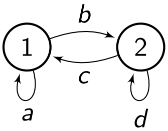

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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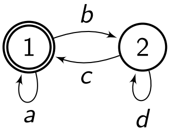

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Büchi-Acceptance

$M^{\omega,k}$ contains infinite paths visiting the first k states infinitely often.

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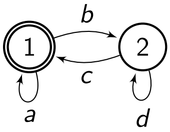

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Büchi-Acceptance

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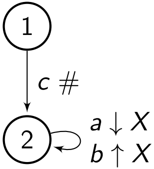
$$M^{\omega,1} = \begin{pmatrix} (a + bd^*c)^\omega \\ d^*c(a + bd^*c)^\omega \end{pmatrix}$$

Pushdown Matrix



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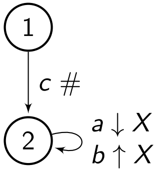

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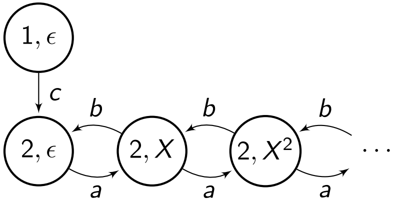

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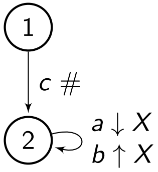

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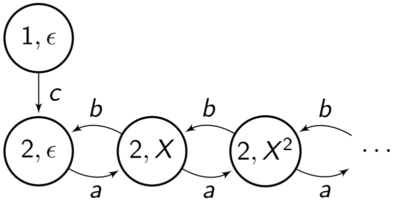

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Intuition: *adjacency matrix* of the graph of configurations of a pushdown automaton

Weighted ω-Pushdown Automata

Definition (Weighted ω-pushdown automaton)

$\mathcal{A} = (n, \Gamma, I, M, k)$ with

- set of states $\{1, \ldots, n\}$, $n \geq 1$,
- pushdown alphabet Γ,
- initial state vector I,
- pushdown matrix M,
- integer k with $0 \leq k \leq n$. $(1, \ldots, k$ are Büchi-accepting states)

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Behavior:

$$\|\mathcal{A}\| = I(M^{\omega,k})_\epsilon$$

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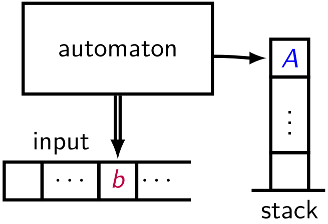

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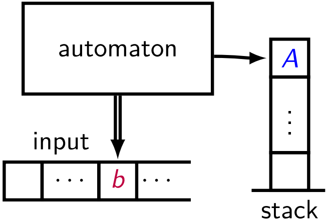

What can it do?

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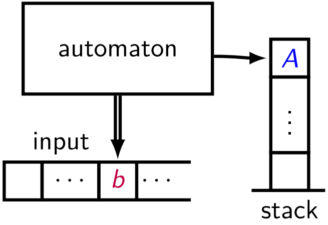

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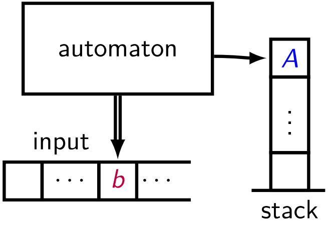

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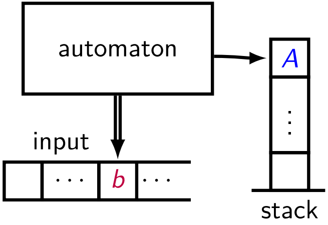

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We use it in a logical characterization.

Summary

Greibach normal form

- can be lifted to ω-algebraic systems

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Results are for $A^{alg}\langle\langle\Sigma^*\rangle\rangle \times A^{alg}\langle\langle\Sigma^\omega\rangle\rangle$

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Thank you for your attention!

Backup

Notes

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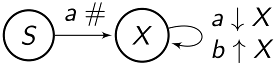

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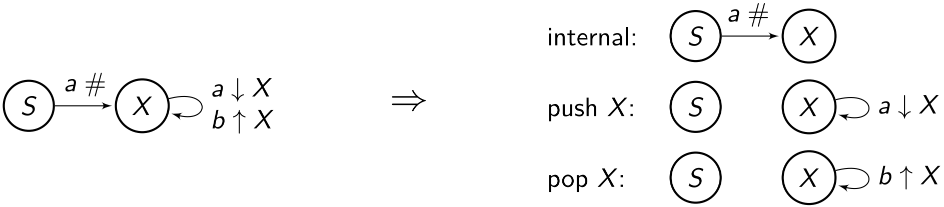

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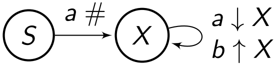

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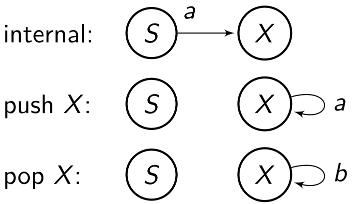

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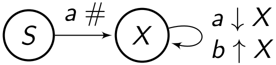

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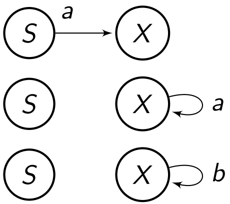

internal: $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$

push X: $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$

pop X: $\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$

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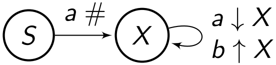

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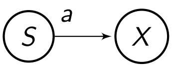
internal: 

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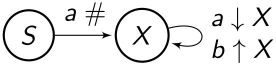

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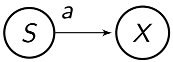
internal: 

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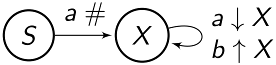

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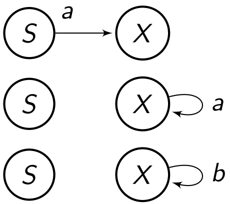

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