Approximate simulation and bisimulation relations for fuzzy automata

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The idea of bisimulations

- ***Simulations*** and ***bisimulations*** - notions used in various settings: modal logic, concurrency theory, set theory, automata theory (fuzzy, weighted, probabilistic, nondeterministic automata).
- They are usually defined as **binary relations** over the domains of two structures, which can be automata.
- Two states of automata are in a relation iff they can perform the same actions to reach bisimilar states.
- When the two structures are the same, the largest bisimulation is an **equivalence relation**, with applications in factorization, separating expressive powers of modal and related logics, concept learning in description logics, etc.

Bisimulations in the literature of weighted automata

Weighted automata (over semirings):

- P. Buchholz. “Bisimulation relations for weighted automata”. In: *Theoretical Computer Science* 393.1 (2008), pp. 109–123
- N. Damljanović, M. Ćirić, and J. Ignjatović. “Bisimulations for weighted automata over an additively idempotent semiring”. In: *Theoretical Computer Science* 534 (2014), pp. 86–100
- S. Stanimirović, A. Stamenković, and M. Ćirić. “Improved algorithms for computing the greatest right and left invariant Boolean matrices and their application”. In: *Filomat* 33.9 (2019), pp. 2809–2831

Bisimulations in the literature of fuzzy automata

Fuzzy automata (over lattice-ordered structures):

- M. Ćirić et al. "Bisimulations for fuzzy automata". In: *Fuzzy Sets and Systems* 186.1 (2012), pp. 100–139
- Y. Cao, G. Chen, and E. E. Kerre. "Bisimulations for Fuzzy-Transition Systems". In: *IEEE Transactions on Fuzzy Systems* 19.3 (2011), pp. 540–552
- L. A. Nguyen and D. X. Tran. "Computing Fuzzy Bisimulations for Fuzzy Structures under the Gödel Semantics". In: *IEEE Transactions on Fuzzy Systems* (2020)

Motivation for approximate bisimulations

- There are cases when there is no bisimulation between two automata, but they are **“more or less”** bisimilar. (Bisimilar **to some degree**.)
- There are cases when no factorization of an automaton can be made via bisimulations, but we want to factorize the automaton, even if the factorized automaton is not strictly language-equivalent to the starting automaton, but equivalent **to some degree**. (Especially of interest for large automata.)
- The presence of the ***vagueness*** in this problem leads to the observation of ***approximate bisimulations*** for **fuzzy automata**.

Heyting algebras, fuzzy sets and fuzzy relations

- $\mathscr{H} = (H, \wedge, \vee, \to, 0, 1)$ - ***Heyting Algebra***, if:
  (H1) $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice,
  (H2) For every $x, y, z \in H$, $x \wedge y \leqslant z$ iff $x \leqslant y \to z$.
  ($x \to y$ is called a ***relative pseudo-complement*** of x in y.)

  $x \leftrightarrow y = (x \to y) \wedge (y \to x)$ - biimplication.

- ***Gödel structure***: $H = [0,1]$, $x \wedge y = \min(x,y)$, $x \vee y = \max(x,y)$,

$$x \to y = \begin{cases} 1, & x \leqslant y \\ y, & \text{otherwise} \end{cases}, \quad x \leftrightarrow y = \begin{cases} 1, & x = y, \\ \min(x,y), & \text{otherwise} \end{cases}.$$

- $\alpha : A \to H$ - a ***fuzzy subset*** of A.
  H^A - the set of all fuzzy subsets of A.
- $\varphi : A \times A \to H$ - a ***fuzzy relation*** on A.
  $H^{A \times A}$ - the set of all fuzzy relations on A.

The degree of subsethood and equality

- $\alpha, \beta \in H^A$ - two fuzzy sets.
- $S(\alpha, \beta) \in H$ - the ***degree of subsethood of*** α ***in*** β:

$$S(\alpha, \beta) = \bigwedge_{a \in A} \alpha(a) \to \beta(a).$$

  $S(\alpha, \beta)$ - a truth degree of "for each $a \in A$: if a belongs to α then a belongs to β."
- $E(\alpha, \beta) \in H$ - the ***degree of equality of*** α ***and*** β:

$$E(\alpha, \beta) = \bigwedge_{a \in A} \alpha(a) \leftrightarrow \beta(a).$$

  $E(\alpha, \beta)$ - a truth degree of "for each $a \in A$: a belongs to α iff a belongs to β."

Approximate simulations

- $\mathcal{A} = (A, \sigma^A, \delta^A, \tau^A)$ and $\mathcal{B} = (B, \sigma^B, \delta^B, \tau^B)$ - fuzzy automata;
- $\lambda \in H$ - a scalar from a complete Heyting algebra (***an approximation degree***).
- A fuzzy relation $\varphi \in H^{A\times B}$ is:

λ-approximate forward simulation, if:

$$S(\sigma^A, \sigma^B \circ \varphi^{-1}) \geqslant \lambda,$$
$$S(\varphi^{-1} \circ \delta^A_x, \delta^B_x \circ \varphi^{-1}) \geqslant \lambda \quad \text{for every } x \in X,$$
$$S(\varphi^{-1} \circ \tau^A, \tau^B) \geqslant \lambda,$$

λ-approximate backward simulation, if:

$$S(\tau^A, \varphi \circ \tau^B) \geqslant \lambda,$$
$$S(\delta^A_x \circ \varphi, \varphi \circ \delta^B_x) \geqslant \lambda \quad \text{for every } x \in X,$$
$$S(\sigma^A \circ \varphi, \sigma^B) \geqslant \lambda.$$

Approximate bisimulations

- A fuzzy relation $\varphi \in H^{A\times B}$ is:

λ-approximate forward bisimulation, if: φ and φ^{-1} are λ-approximate forward simulations,

λ-approximate forward-backward bisimulation (λ-FB), if:

$$E(\sigma^A, \sigma^B \circ \varphi^{-1}) \geqslant \lambda,$$
$$E(\varphi^{-1} \circ \delta_x^A, \delta_x^B \circ \varphi^{-1}) \geqslant \lambda \quad \text{for every } x \in X,$$
$$E(\varphi^{-1} \circ \tau^A, \tau^B) \geqslant \lambda,$$

λ-approximate backward bisimulation, if: φ and φ^{-1} are λ-approximate backward simulations,

λ-approximate backward-forward bisimulation (λ-BF), if:

$$E(\tau^A, \varphi \circ \tau^B) \geqslant \lambda,$$
$$E(\delta_x^A \circ \varphi, \varphi \circ \delta_x^B) \geqslant \lambda \quad \text{for every } x \in X,$$
$$E(\sigma^A \circ \varphi, \sigma^B) \geqslant \lambda.$$

An example

Let $\mathscr{A} = (A, \sigma^A, \delta^A, \tau^A)$ and $\mathscr{B} = (B, \sigma^B, \delta^B, \tau^B)$ be fuzzy automata over the Gödel structure $\mathscr{H}$ and $X = \{x, y\}$ with $|A| = 3, |B| = 2$ and

$$\sigma^A = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad \tau^A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}},$$

$$\delta_x^A = \begin{bmatrix} 1 & 0.3 & 0.4 \\ 0.5 & 1 & 0.3 \\ 0.4 & 0.6 & 0.7 \end{bmatrix}, \quad \delta_y^A = \begin{bmatrix} 0.5 & 0.6 & 0.2 \\ 0.6 & 0.3 & 0.4 \\ 0.7 & 0.7 & 1 \end{bmatrix},$$

$$\sigma^B = \begin{bmatrix} 0.8 & 0.8 \end{bmatrix}, \quad \tau^B = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}},$$

$$\delta_x^B = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 0.7 \end{bmatrix}, \quad \delta_y^B = \begin{bmatrix} 0.6 & 0.6 \\ 0.7 & 1 \end{bmatrix}.$$

An example (cont.)

- For $\lambda = 1$ - no simulations nor bisimulations;
- $\lambda = 0.8$:

$$\varphi^{\mathrm{FS}} = \begin{bmatrix} 1 & 0.7 \\ 1 & 0.7 \\ 0.6 & 1 \end{bmatrix}, \quad \varphi^{\mathrm{BS}} = \begin{bmatrix} 1 & 0.7 \\ 1 & 0.7 \\ 0.7 & 1 \end{bmatrix}.$$

- $\lambda = 0.7$:

$$\psi^{\mathrm{FS}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0.6 & 1 \end{bmatrix}, \quad \psi^{\mathrm{BS}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \psi^{\mathrm{BFB}} = \begin{bmatrix} 1 & 0.6 \\ 1 & 0.6 \\ 1 & 1 \end{bmatrix}.$$

Theorem

Let $\mathscr{A}$ and $\mathscr{B}$ be fuzzy automata, $\lambda \in H$, and $\varphi \in H^{A\times B}$. Then:

1. *If φ is **λ-approximate simulation**, then*
$$S([[\mathscr{A}]],[[\mathscr{B}]]) \geqslant \lambda,$$
2. *If φ is **λ-approximate bisimulation**, then*
$$E([[\mathscr{A}]],[[\mathscr{B}]]) \geqslant \lambda.$$

Theorem

*Let $\mathscr{A}$ and $\mathscr{B}$ be fuzzy automata such that there exists **at least one** λ-approximate (bi)simulation between $\mathscr{A}$ and $\mathscr{B}$. Then there exists **the greatest** λ-approximate (bi)simulation, which is also a **partial fuzzy function.***

Theorem

Let $\mathcal{A}$ be a fuzzy automaton and $\lambda \in H$. Then:

- *There exists **the greatest** λ-approximate (bi)simulation on $\mathcal{A}$,*
- *The greatest λ-approximate **simulation** on $\mathcal{A}$ is a **fuzzy quasi-order** on A,*
- *The greatest λ-approximate **bisimulation** on $\mathcal{A}$ is a **fuzzy equivalence** on A.*
- *If φ is λ-approximate bisimulation fuzzy equivalence on $\mathcal{A}$, or λ-approximate backward simulation fuzzy quasi-order on $\mathcal{A}$, then*

$$E([\![\mathcal{A}]\!], [\![\mathcal{A}/\varphi]\!]) \geqslant \lambda.$$

An example

Let $\mathscr{H}$ be the Gödel structure, and let $\mathscr{A} = (A, \sigma^A, \delta^A, \tau^A)$ be a fuzzy automaton over $\mathscr{H}$ and $X = \{x\}$ with $|A| = 6$ and

$$\sigma^A = \begin{bmatrix} 0.2 & 0.7 & 0.4 & 0.9 & 0.7 & 0.7 \end{bmatrix},$$

$$\tau^A = \begin{bmatrix} 0.6 & 0.8 & 0.8 & 0.7 & 0.8 & 0.6 \end{bmatrix}^{\mathrm{T}},$$

$$\delta_x^A = \begin{bmatrix} 0.4 & 1 & 0 & 0.6 & 0.2 & 1 \\ 0.5 & 0.5 & 0.8 & 0.8 & 0.4 & 0.9 \\ 0.6 & 0.6 & 0.2 & 0.2 & 0.4 & 0.7 \\ 0 & 0.5 & 1 & 0.9 & 0.2 & 0.9 \\ 0.4 & 0.9 & 0.9 & 0.5 & 0.8 & 0.4 \\ 0.7 & 0.8 & 0.2 & 0.4 & 0 & 0.2 \end{bmatrix}.$$

$$[\![\mathscr{A}]\!](u) = \begin{cases} 0.7, & \text{if } u = e, \\ 0.8, & \text{otherwise.} \end{cases}$$

An example (cont.)

- Case $\lambda = 1$: ***No reduction*** can be made via any type of bisimulation.
- Case $\lambda = 0.7$. Then the greatest ***λ-approximate backward simulation***:

$$\psi = \psi^{\mathbf{BS}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.2 & 1 & 0.4 & 1 & 1 & 1 \\ 0.2 & 1 & 1 & 1 & 1 & 1 \\ 0.2 & 1 & 0.4 & 1 & 1 & 1 \\ 0.2 & 1 & 0.4 & 1 & 1 & 1 \\ 0.2 & 1 & 0.4 & 1 & 1 & 1 \end{bmatrix}$$

which means that $\mathscr{A}/\psi$ has **3 different states**. It can be verified that $[\![\mathscr{A}/\psi]\!](u) = 0.8$, for every $u \in X^*$.

An example (cont.)

- Case $\lambda = 0.7$. Then the greatest ***λ-approximate forward bisimulation***

$$\varphi = \varphi^{\mathbf{FB}} = \begin{bmatrix} 1 & 0.6 & 0.6 & 0.6 & 0.6 & 1 \\ 0.6 & 1 & 0.6 & 1 & 0.6 & 0.6 \\ 0.6 & 0.6 & 1 & 0.6 & 0.6 & 0.6 \\ 0.6 & 1 & 0.6 & 1 & 0.6 & 0.6 \\ 0.6 & 0.6 & 0.6 & 0.6 & 1 & 0.6 \\ 1 & 0.6 & 0.6 & 0.6 & 0.6 & 1 \end{bmatrix},$$

  which means that $\mathcal{A}/\varphi$ has **4 different states**. It can be verified that $[\![\mathcal{A}/\varphi]\!](u) = 0.8$, for every $u \in X^*$.
- There are ***no reductions*** via other types of λ-approximate bisimulations.

Conclusion and future work

- More details on approximate bisimulations:
  S. Stanimirović, I. Micić, and M. Ćirić. "Approximate Bisimulations for Fuzzy Automata over Complete Heyting Algebras". In: *IEEE Transactions on Fuzzy Systems* (2020)
- Future work:
  - Approximate (bi)simulations which are partial fuzzy functions and uniform fuzzy relations,
  - Faster algorithms for computing the greatest approximate (bi)simulations,
  - Approximate (bi)simulations for fuzzy/weighted automata over other structures.

Thank you for your attention!