Parameter Synthesis for One-Counter Automata

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joint work with Ritam Raha

April 2021

Static reachability analysis via the control flow graph

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1  x = 0
2  i = 0
3  i += x
4  while i >= 0:
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6          print("Hello world!")
7      if i == 1:
8          print("Lockdown + babies = pain")
9      if i == 2:
10         print("Please, help me!")
11     if i >= 3:
12         assert(False)
13     i -= 1
14 # end program
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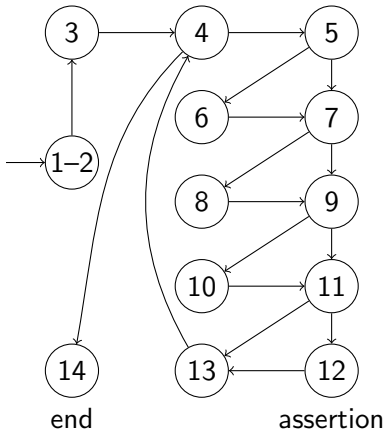

Extending the CFG with a counter

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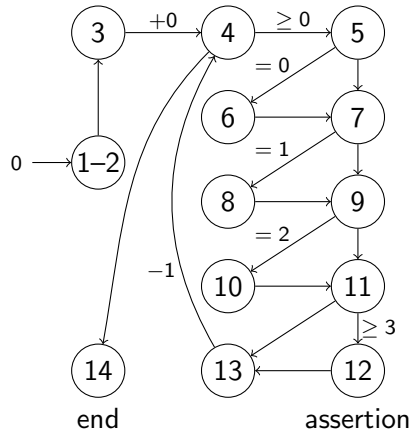

Parametric one-counter automata

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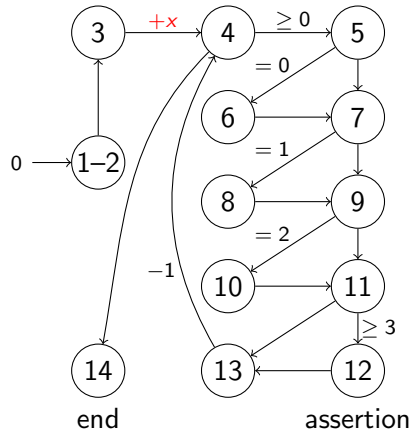
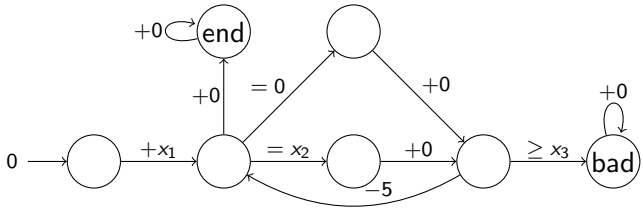


Outline

Parametric One-Counter Automata

Natural-valued parameters

$X = \{x_1, \ldots, x_n\}$



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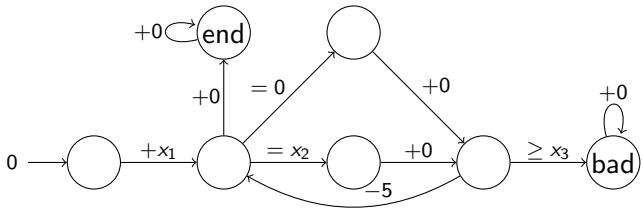

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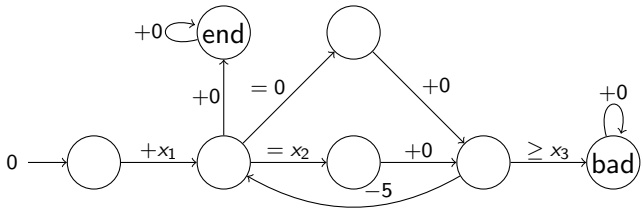

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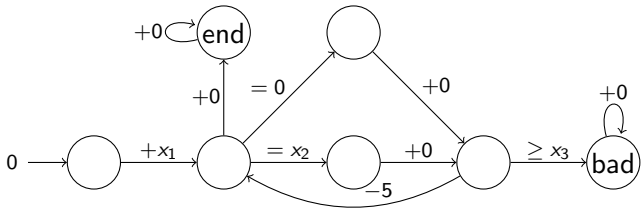

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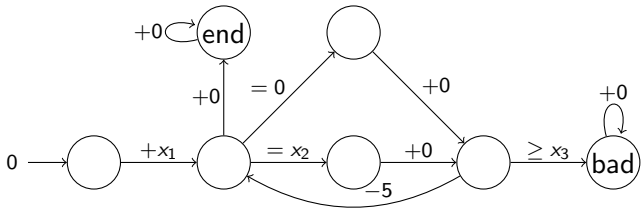

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! Runs are restricted to keeping the counter value non-negative

Synthesis problems

Definition (Parameter-value synthesis)

Is there some valuation $V : X \to \mathbb{N}$ such that all (infinite) runs of $\mathcal{A}$ satisfy a given ω-regular property?

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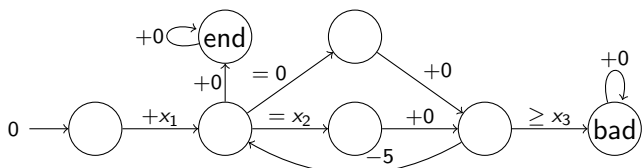

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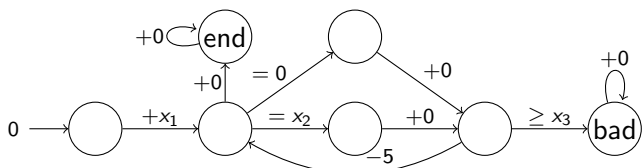

	LTL	Reachability	Safety	Büchi	coBüchi
Lower bound	PSPACE-hard	coNP-hard	— NP^{NP}-hard —		
Upper bound	in N3EXP	— in N2EXP —			

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Theorem (Lipshitz'78, Lechner-Ouaknine-Worrell'15)

The existential fragment of PAD (EPAD) is decidable and in NEXP.

A restricted one-alternation fragment: $\forall\exists_R$PAD

- $\forall\exists_R\mathrm{PAD} = \forall z_1 \ldots \forall z_n \exists x_1 \ldots \exists x_m . \varphi(\boldsymbol{x}, \boldsymbol{z})$
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Claim (Bozga-Iosif'05, Lechner'15)

The synthesis problems for SOCAP are decidable via an encoding into $\forall\exists_R\text{PAD}^+$.

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Encoding the safety synthesis problem

Safety synthesis

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Encoding reachability [Lechner'15]

- PAD formula $\Phi(\boldsymbol{x}) = \exists \boldsymbol{f} \bigvee_i \bigwedge_j g_j(\boldsymbol{x}) \mid h_j(\boldsymbol{f}, \boldsymbol{x}) \wedge \varphi_i(\boldsymbol{x}) \wedge \boldsymbol{f}, \boldsymbol{x} \geq 0$
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Constructing the formula [Haase et al.'09]

Boolean combination of sub-formulas for 3 subcases: reach t via a run that

- has no positive cycles
- ...

$$\exists \boldsymbol{f} \bigvee_i \varphi^{(i)}_{\text{flow}}(\boldsymbol{f}) \wedge \varphi^{(i)}_{\text{weight}}(\boldsymbol{f}, \boldsymbol{x}) \wedge \varphi^{(i)}_{\text{nopos}}(\boldsymbol{x})$$

The flow sub-formula

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Path formulas

V should satisfy φ_{flow} iff $\mathcal{A}$ has a path reaching t from s

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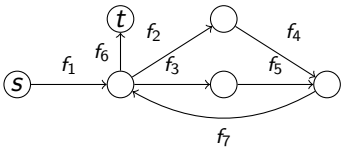

$$\bigwedge_{q \notin \{s,t\}} \left(\sum_{p \in T(\cdot,q)} f(p,q) = \sum_{r \in T(q,\cdot)} f(q,r) \right) \quad (1)$$

Theorem (Euler's theorem for digraphs)

There is an s–t path iff there is a valuation of the f_i such that

- *the subgraph induced by edges with nonzero flow and $\{(t,s)\}$ is strongly connected,*
- *it satisfies (1) and $\sum_{p \in T(\cdot,s)} f(p,s) - \sum_{r \in T(s,\cdot)} f(s,r) = 1$.*

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A formula per subgraph

Each $\varphi^{(i)}_{\text{flow}}(\boldsymbol{f})$ captures the flow constraints over one subgraph that satisfies the support condition.

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No positive cycles

Now $\varphi^{(i)}_{\text{nopos}}(\boldsymbol{x})$ becomes easy: a conjunction of weight constraints over all simple cycles in the subgraph.

The weight sub-formula: from paths to runs

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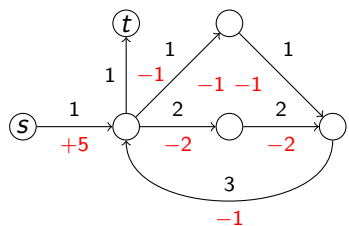
A path is not a run

Not every path can be lifted to a run because:

- of equality tests,
- the lower-bound tests,
- the counter value cannot go negative



Decomposed flows

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Decomposed reachability certificates

Instead of a single flow f, we want $f_1, f_2, \ldots, f_{|Q|}$ such that $f = \sum_{i=1}^{|Q|} f_i$ and:

- f_i is a flow witnessing a q_i–q_{i+1} path,
- $f_j(p, q_i) = 0$ for all $i \leq j$ and all $(p, q_i) \in T$.

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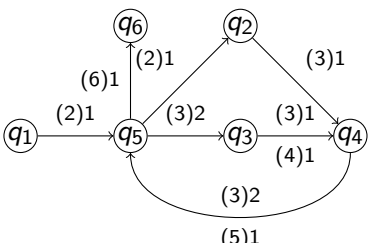

Case 1: No positive cycles

To check the path can be lifted to a run:

$$\bigwedge_{m=1}^{|Q|} \sum_{i=1}^{m} \sum_{t \in T} f_i(t)\delta(t) \geq 0$$

Weight formulas using divisibility

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A formula per decomposition

Each $\varphi^{(i)}_{\text{weight}}(\boldsymbol{x}, \boldsymbol{f})$ captures the weight constraints over one decomposition of the flow.

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Divisibility to the rescue

We replace every appearance of fx with a product variable z_{fx}. Then the weight constraints can be rewritten:

$$(x_i \mid z_{f_i x_i}) \wedge (x_i > 0 \leftrightarrow z_{f_i x_i} > 0) \wedge \left(\sum z_{f_i x_i} \geq 0 \right)$$

Final PAD encoding

$$\exists \boldsymbol{f} \bigvee_i \bigwedge_j g_j(\boldsymbol{x}) \mid h_j(\boldsymbol{f}, \boldsymbol{x}) \wedge \varphi_i(\boldsymbol{x}) \wedge \boldsymbol{f}, \boldsymbol{x} \geq 0$$

The safety synthesis problem

Consider an instance of the safety synthesis problem.

- It has a positive answer iff $\forall \boldsymbol{x}.\Phi(\boldsymbol{x})$ is false
- $\forall \boldsymbol{x}.\Phi(\boldsymbol{x})$ is a sentence in $\forall\exists_R \mathrm{PAD}^+$

A restricted one-alternation fragment: $\forall\exists_R$PAD

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$\forall\exists_R \mathrm{PAD}^+$ is undecidable

Lemma (Lechner'15)

$\forall\exists_R PAD \equiv \forall\exists_R PAD^+$

Sketch: We can rewrite $\neg(a \mid b)$ using the identity

$$\neg(a \mid b) \iff \exists q \exists r.(b = aq + r) \wedge (0 < r < b)$$

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Theorem (Bozga-Iosif'05)

Hilbert's 10th problem can be encoded in $\forall\exists_R PAD$, hence it is undecidable.

Sketch: Using the single restricted alternation we define

1. LCM
2. Square (x^2)
3. Multiplication

A stronger restriction of $\forall\exists_R$PAD

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- The Bozga-Iosif-Lechner fragment of $\forall\exists_R$PAD:

$$\forall \boldsymbol{z} \exists \boldsymbol{x} \bigvee_{i \in I} \bigwedge_{j \in J_i} (g_j(\boldsymbol{z}) \mid h_j(\boldsymbol{x}, \boldsymbol{z}) \wedge g_j(\boldsymbol{z}) \neq 0) \wedge \varphi_i(\boldsymbol{z}) \wedge \boldsymbol{x}, \boldsymbol{z} \geq 0$$

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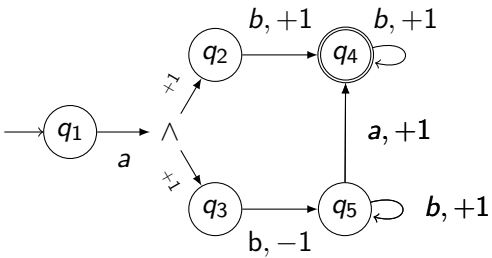
We massage $\forall \boldsymbol{x}.\Phi(\boldsymbol{x})$ to get it into the BIL-fragment.

Corollary

The safety synthesis problem is decidable and in N2EXP.

Outline

Alternating two-way automata (A2A)



- existential and universal non-determinism
- a run is a tree
- accepting if all infinite branches are Büchi-accepting

Theorem (Serre'06)

The emptiness problem for A2As is in PSPACE.

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For every SOCAP $\mathcal{A}$ we construct an A2A $\mathcal{T}$ of poly-size which accepts words corresponding to valuations for which universal reachability holds.

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- For every transition we build an A2A

From SOCAP to A2A

$Op = \{-1, 0, +1\} \cup ZT \cup PT$

Lemma (Based on Bollig-Quaas-Sangnier'19)

For every SOCAP $\mathcal{A}$ we construct an A2A $\mathcal{T}$ of poly-size which accepts words corresponding to valuations for which universal reachability holds.

Sketch:

- Encode valuations as parameter words
- For every transition we build an A2A
- Accept the reaching runs
- (. . . and runs that "die off")

SOCAP to A2A

Valuations as words

- $V : X \to \mathbb{N}$ encoded with $\Sigma = X \cup \{\square\}$

SOCAP to A2A

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SOCAP to A2A

Valuations as words

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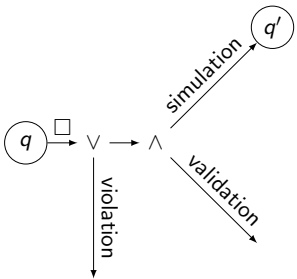
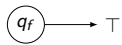
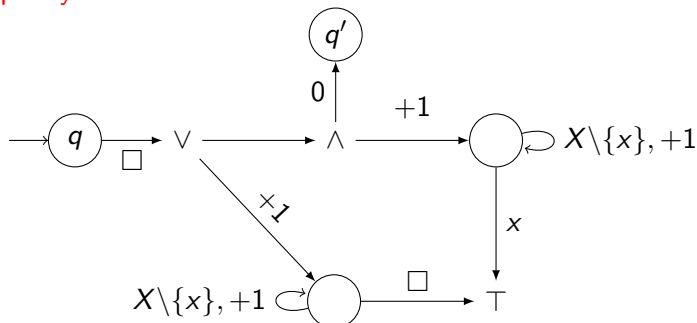
A2As for each transition (q, q')





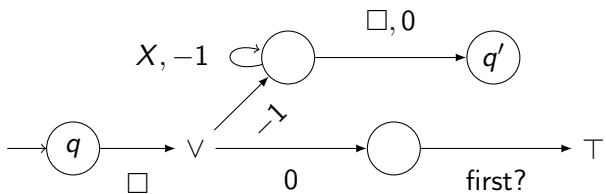
SOCAP to A2A: Equality tests

Encoding an equality test



SOCAP to A2A: Decrements

Encoding a decrement



SOCAP to A2A: Putting things together

To check all runs:

If we created subautomata and transitions

- $(q, \square, \mathcal{T}_1)$ to simulate $q \xrightarrow{op_1} q_1$,
- $(q, \square, \mathcal{T}_2)$ to simulate $q \xrightarrow{op_2} q_2$.

Then, we replace the transitions with: $(q, \square, \mathcal{T}_1 \wedge \mathcal{T}_2) \in \mathcal{T}$

SOCAP to A2A: An algorithm for reachability

Lemma

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- For every A2A $\mathcal{T}$ there is an NWA $\mathcal{B}$ of exponential size accepting same language
- Non-emptiness witnesses for NWAs are simple "lassos"
- $\implies$ the reachability synthesis problem admits exponential (w.r.t. the SOCAP) witnesses and thus polynomial in binary encoding

SOCAP to A2A: An algorithm for reachability

Lemma

The reachability synthesis problem for SOCAP with non-parametric updates is in PSPACE.

- For every A2A $\mathcal{T}$ there is an NWA $\mathcal{B}$ of exponential size accepting same language
- Non-emptiness witnesses for NWAs are simple "lassos"
- $\implies$ the reachability synthesis problem admits exponential (w.r.t. the SOCAP) witnesses and thus polynomial in binary encoding
- Guess a valuation and check universal reachability for resulting SOCA

Theorem

The reachability synthesis problem for SOCAP is in $\mathrm{NP}^{\mathrm{coNP}}$.

Conclusion

	LTL	Reachability	Safety	Büchi	coBüchi
Lower bound	PSPACE-hard	coNP-hard	— NP^{NP}-hard —		
Upper bound	in N3EXP	— in N2EXP —			

Open questions

- What is the exact complexity of the synthesis problems?
- What about approximations via continuous relaxations?
- What is the complexity of the BIL fragment?
- Can the A2A approach be extended to handle parametric updates?
  - by adding a single pebble?