Kleene and Büchi for Weighted Forest Automata over M-Monoids

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QuantLA

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1 Introduction

Structure of the Talk

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Structure of the Talk

- WTA over M-Monoids

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- Weighted Forest Automata

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An operation is a map $\omega \colon M^k \to M$

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M-Monoids generalise semirings $(S, +, \cdot, 0, 1)$

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M-Monoids are more general than semirings

e.g. average, weighted sums, …

2 WTA over M-monoids

$$A = (Q, n, F, \mu, \nu)$$

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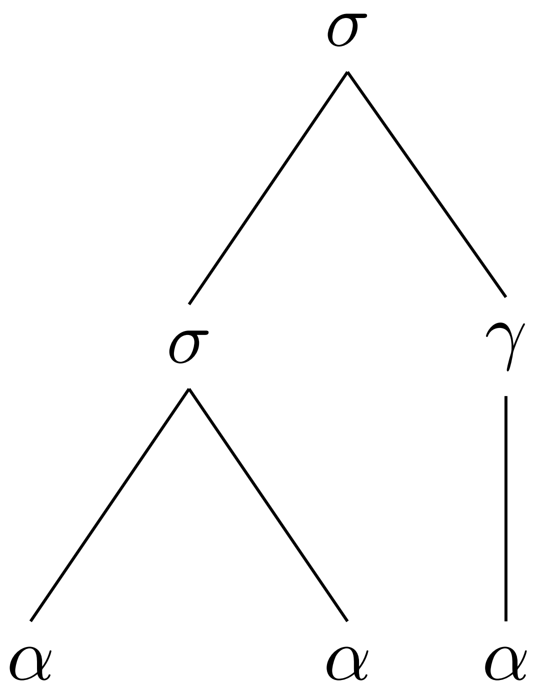
WTA over semirings



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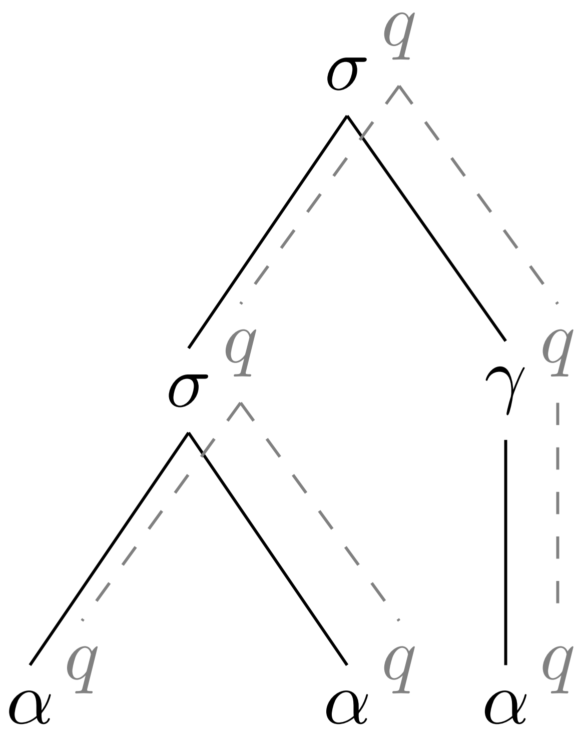

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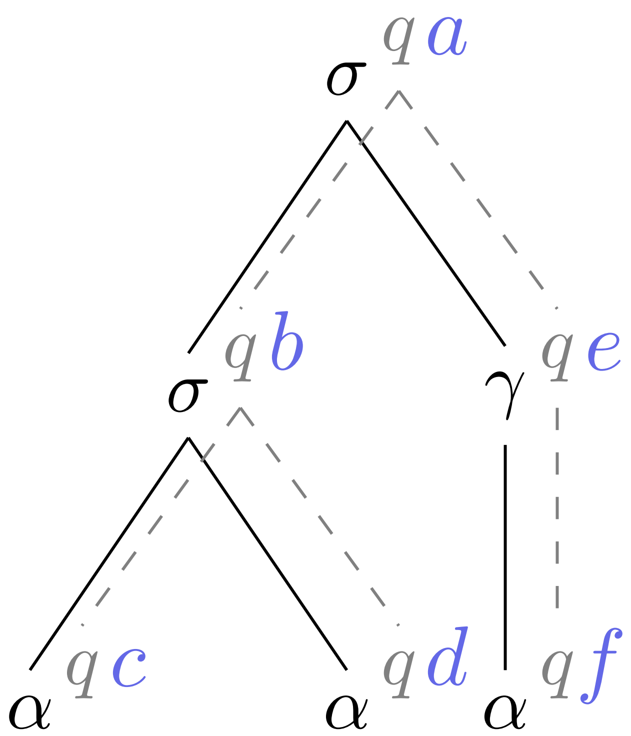

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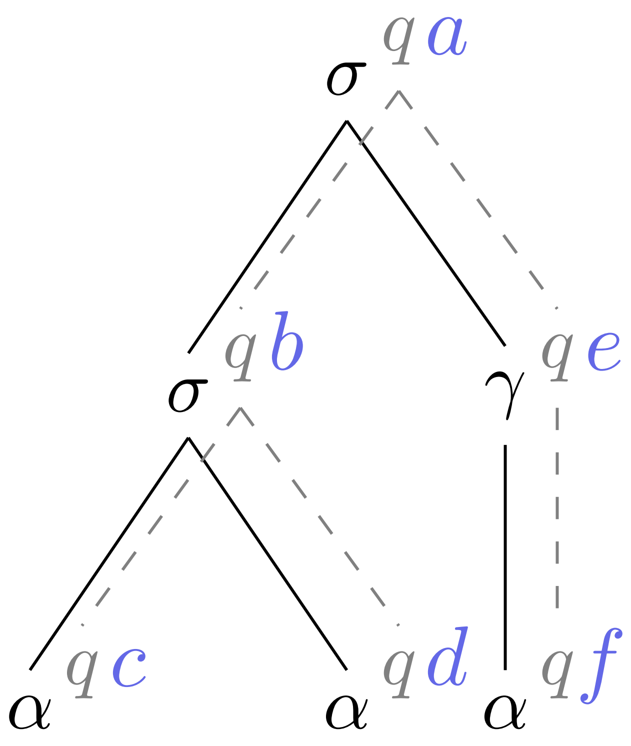

$$a \cdot b \cdot c \cdot d \cdot e \cdot f$$

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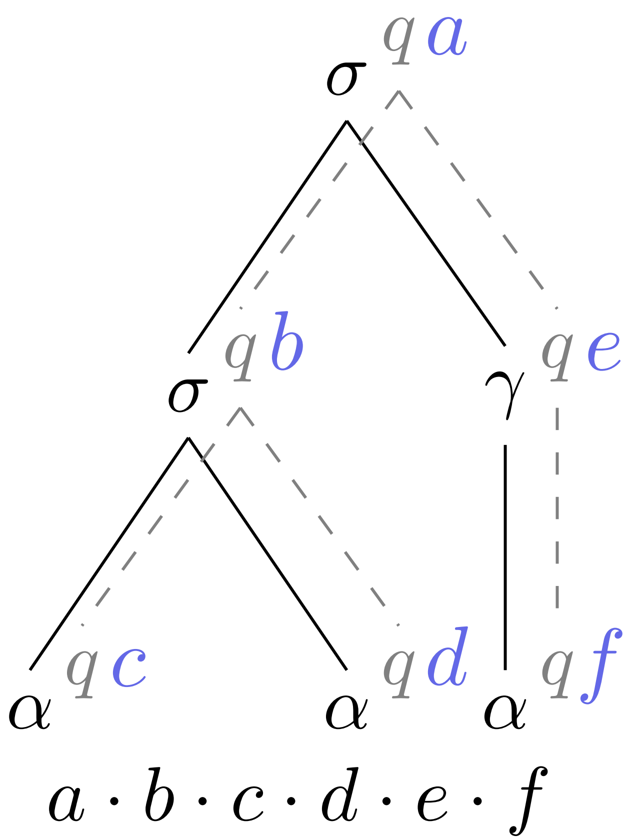

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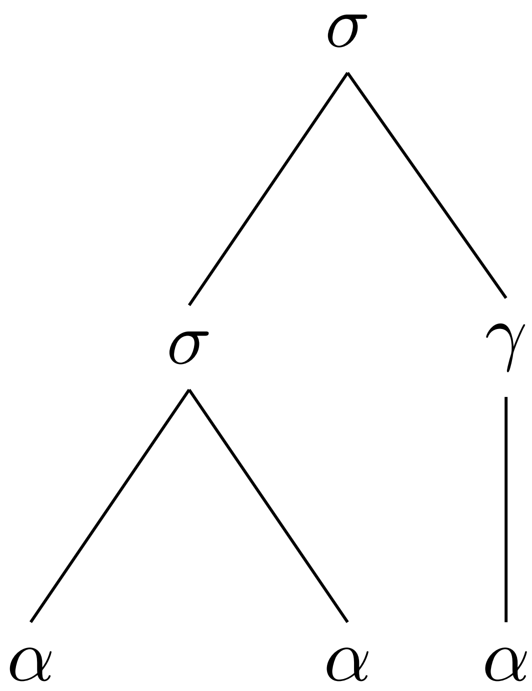

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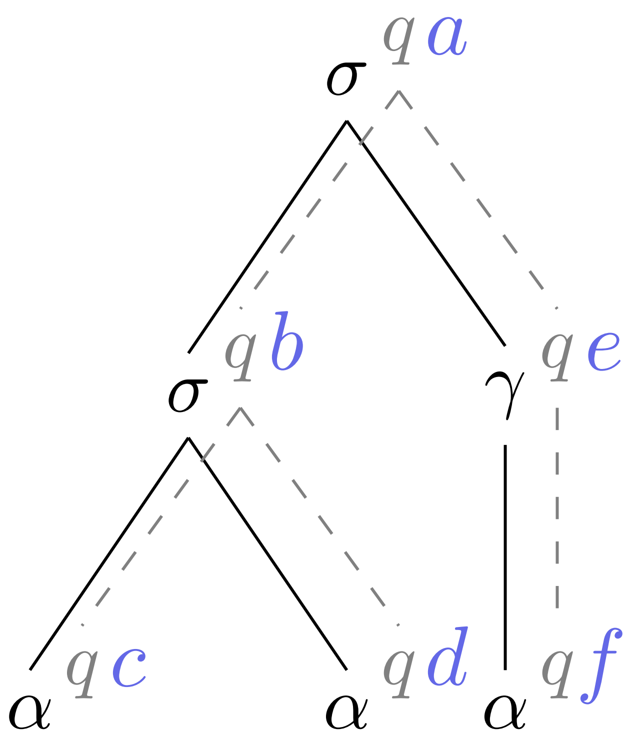

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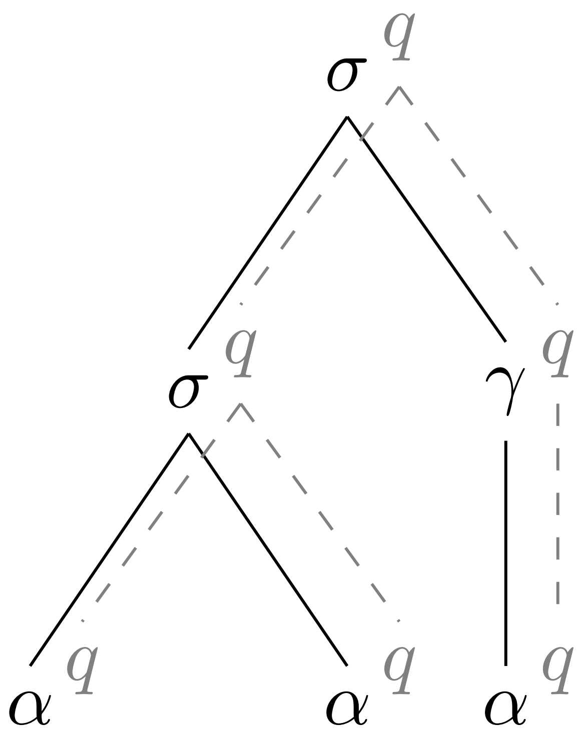

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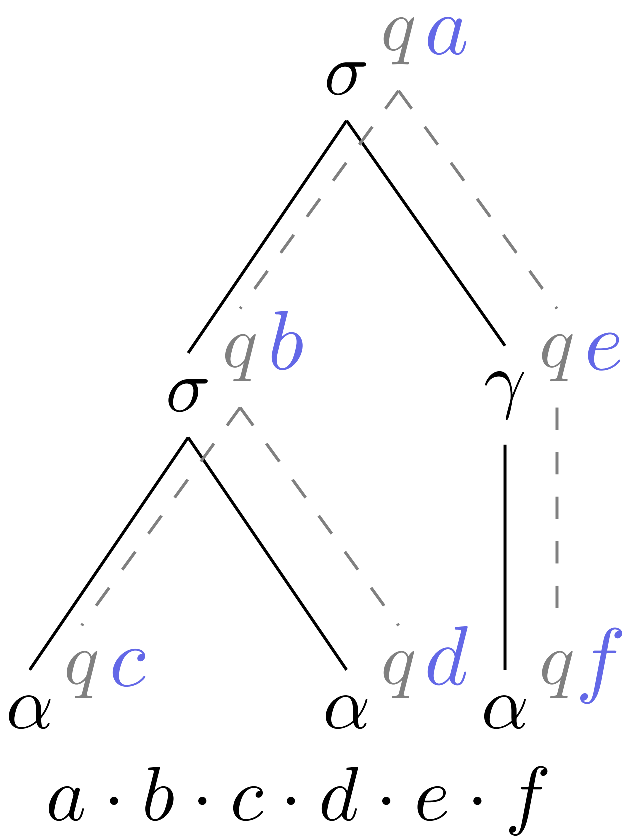

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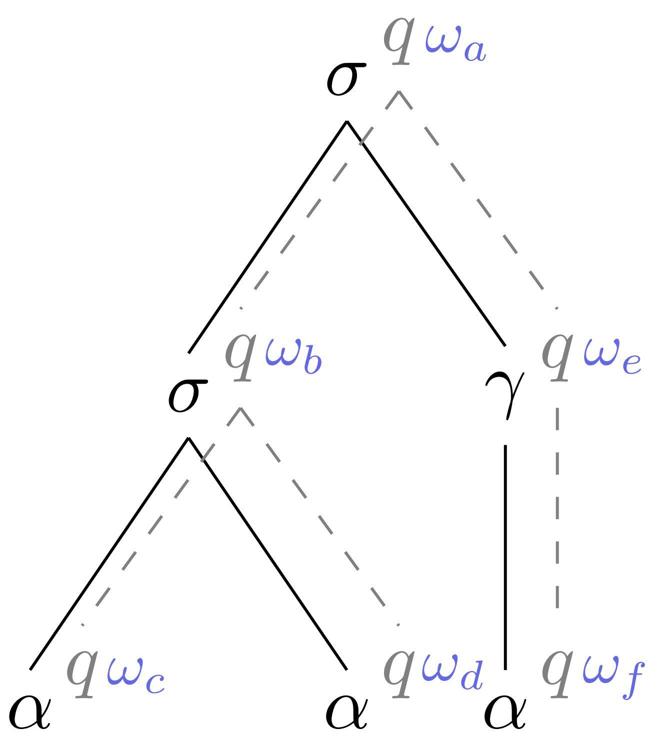

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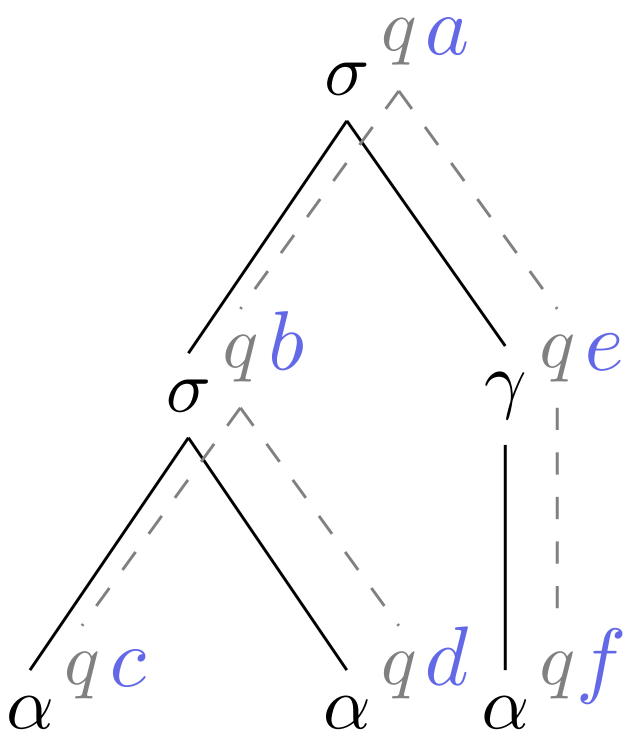
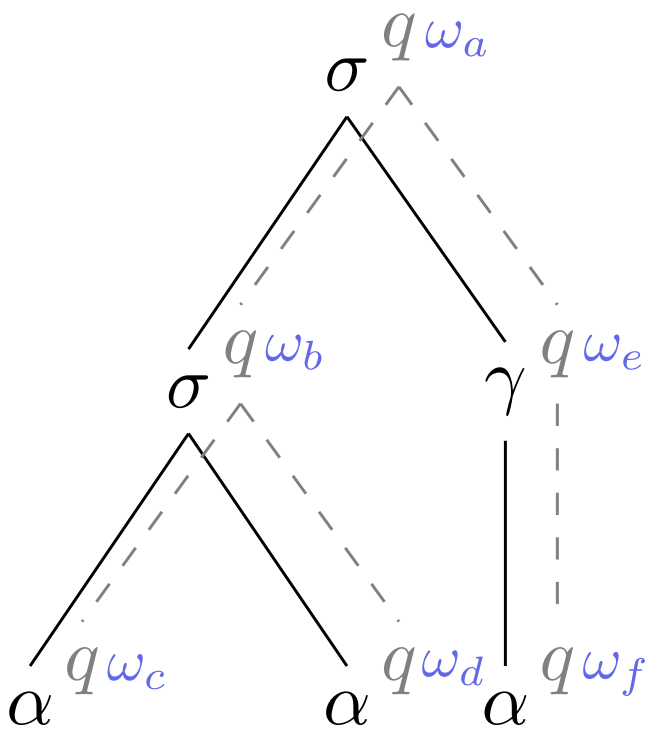
WTA over semirings



$a \cdot b \cdot c \cdot d \cdot e \cdot f$

WTA over M-monoids



$\omega_a(\omega_b(\omega_c, \omega_d), \omega_e(\omega_f))$

3 Forests

Σ – ranked alphabet

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$\mathrm{T}_\Sigma(X_n)$ trees over Σ with variables in X_n

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Set of (m,n) - **forests** over Σ

$$\mathrm{F}(\Sigma)_n^m := \{n\} \times \mathrm{T}_\Sigma(X_n)^m$$

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m-tuples of trees with variables in X_n

3 Weighted Forest Automata

Σ – ranked alphabet
M – M-monoid

A **weighted forest automaton** over Σ and M is a tuple

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where

Q alphabet of **states**

m **width** of forests

n number of **variables**

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$$A = (Q, m, n, F, \mu, \nu)$$

where

$$F = (F_1, \ldots, F_m)$$

$$F_i : Q \to \Omega_M^{(1)}$$

root distribution function

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$$\mu = (\mu_k \colon Q^k \times \Sigma^{(k)} \times Q \to \Omega_M^{(k)} \mid k \geq 0)$$

transition mappings

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variable assignment

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$$\left(1, \qquad\qquad , \qquad\qquad \right)$$

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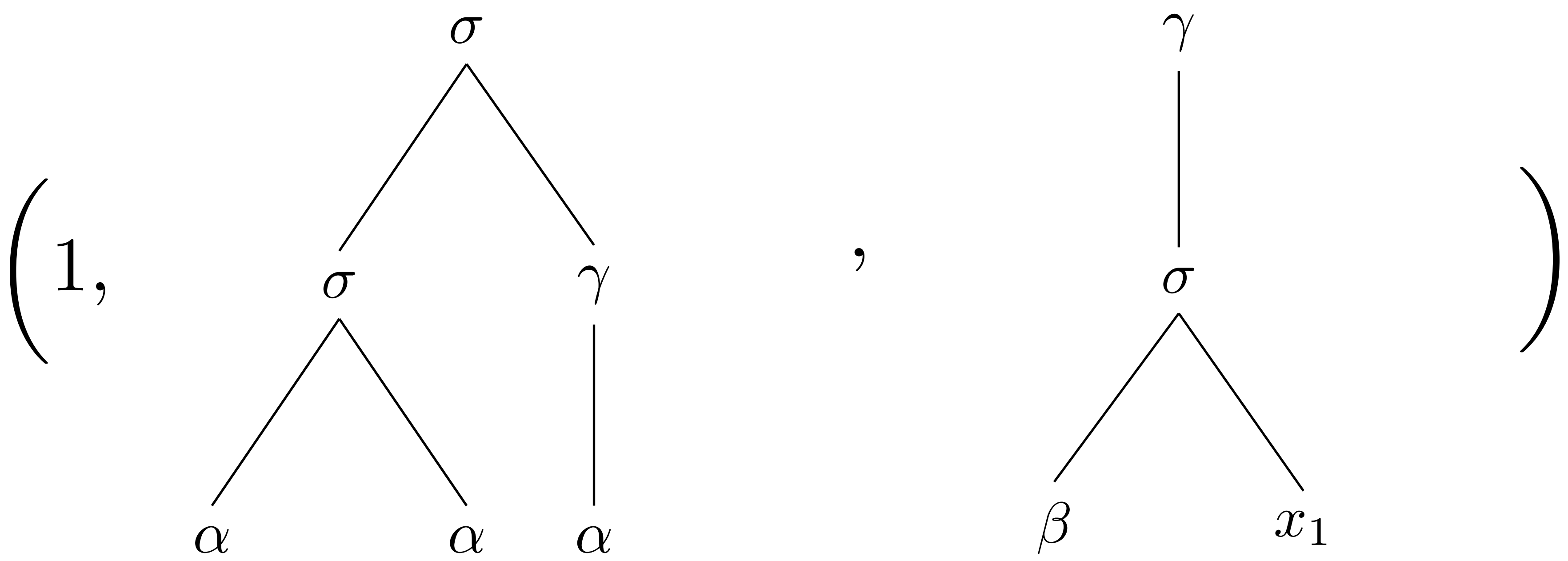

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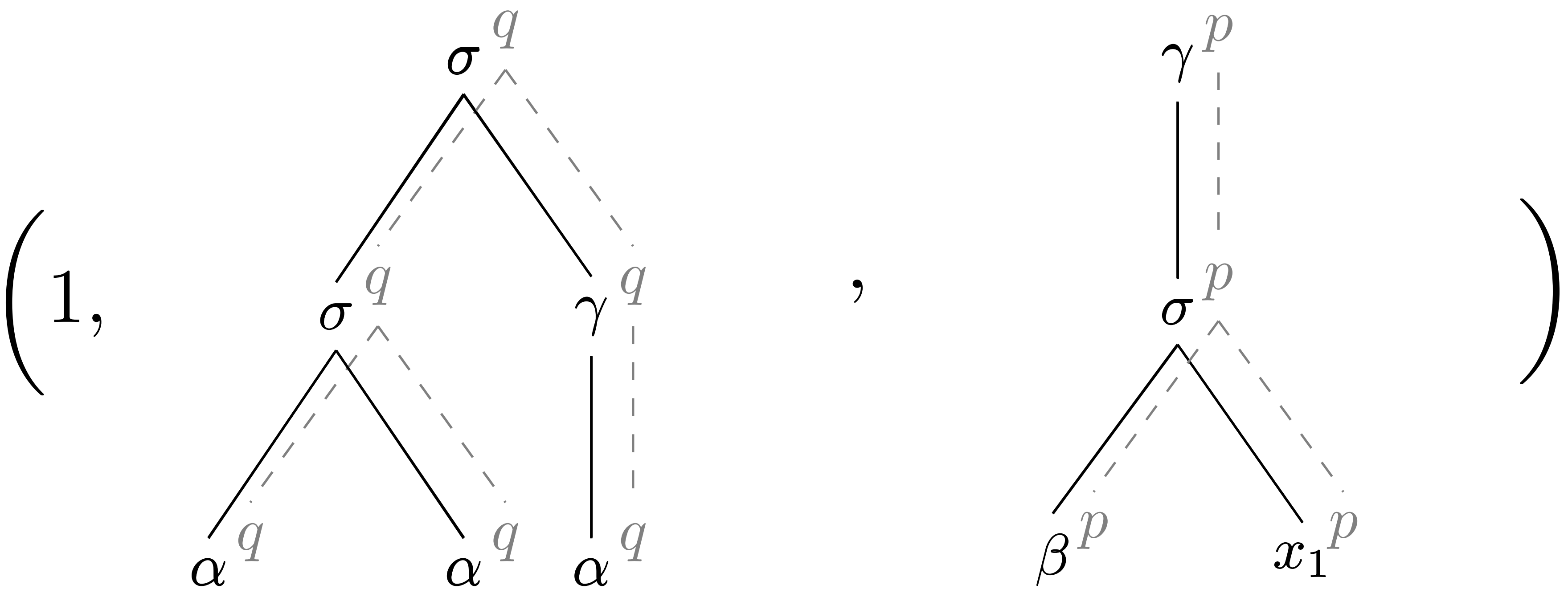

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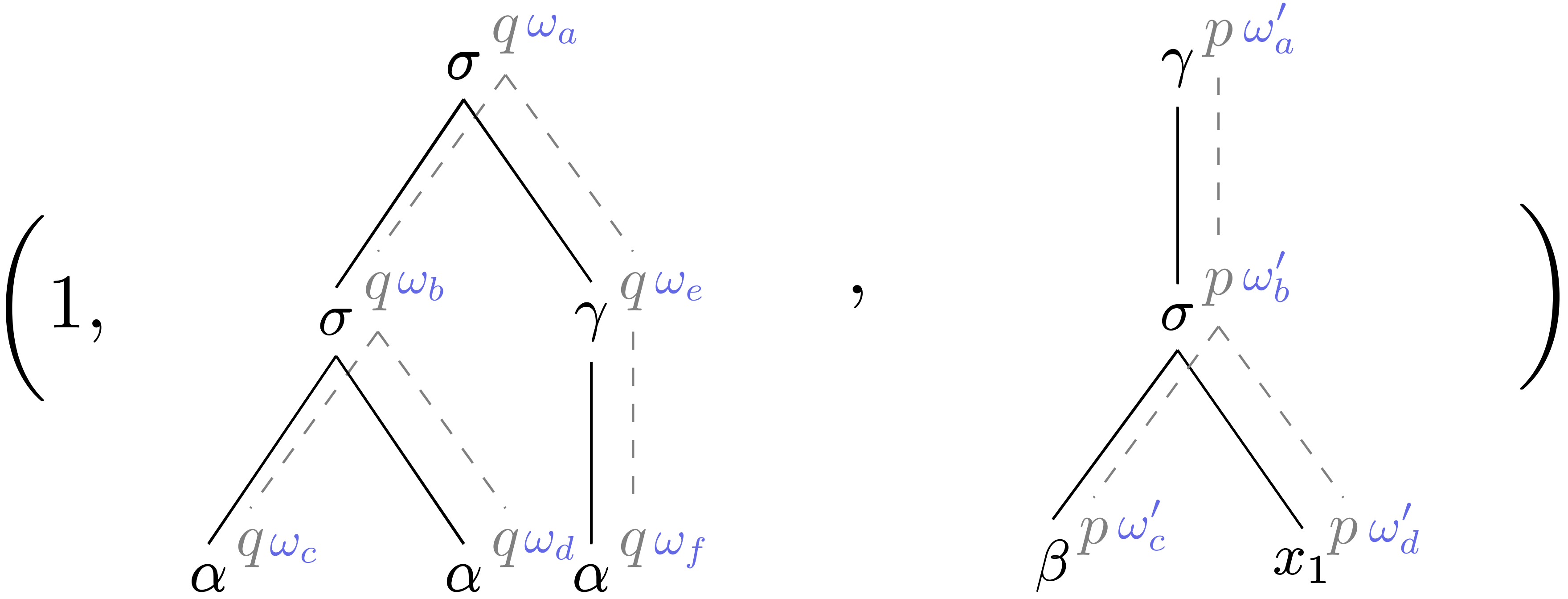

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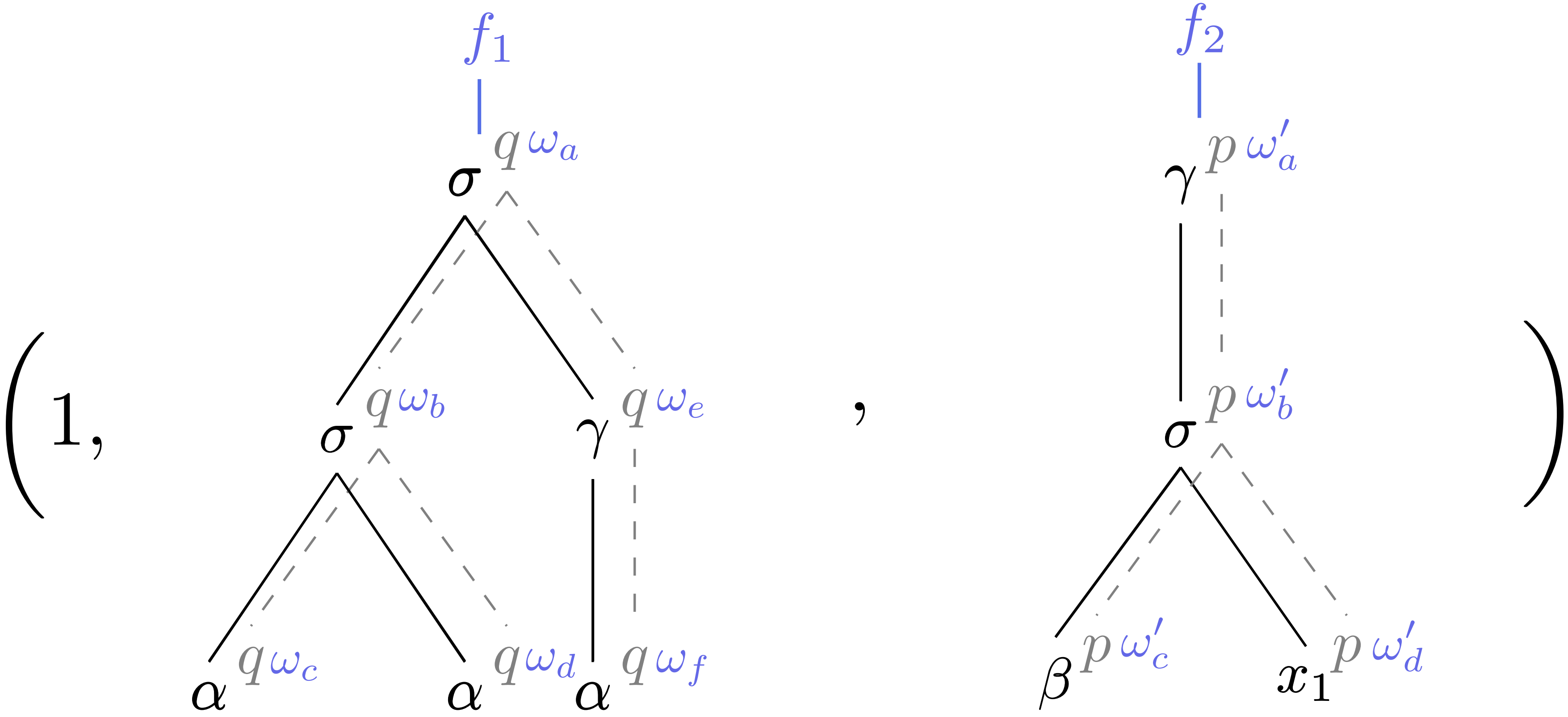

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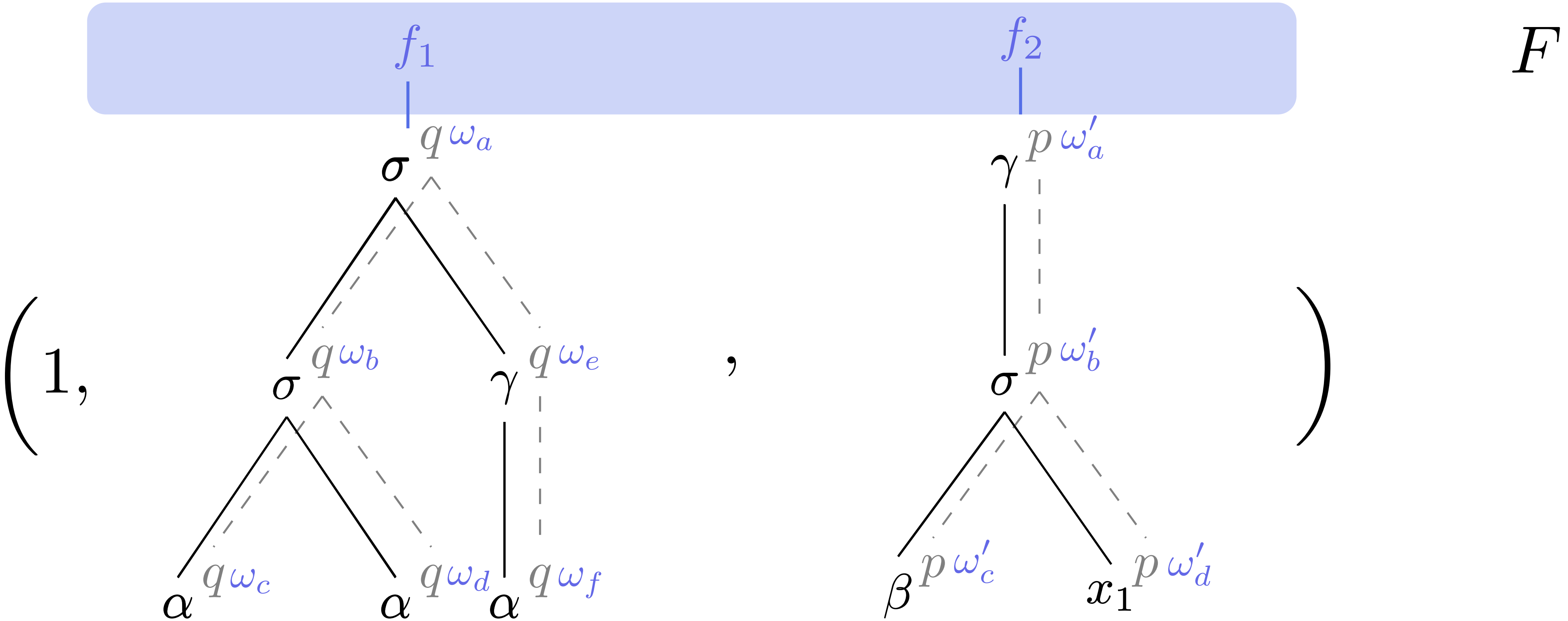

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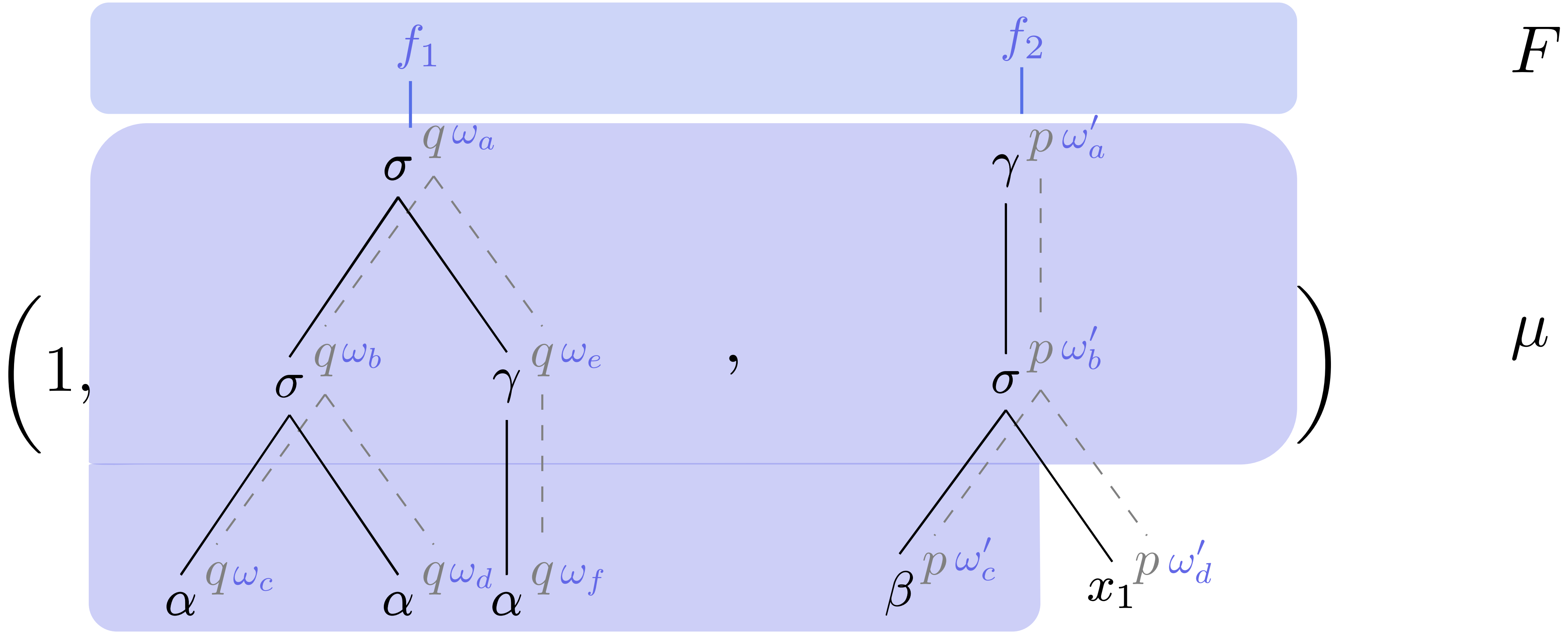

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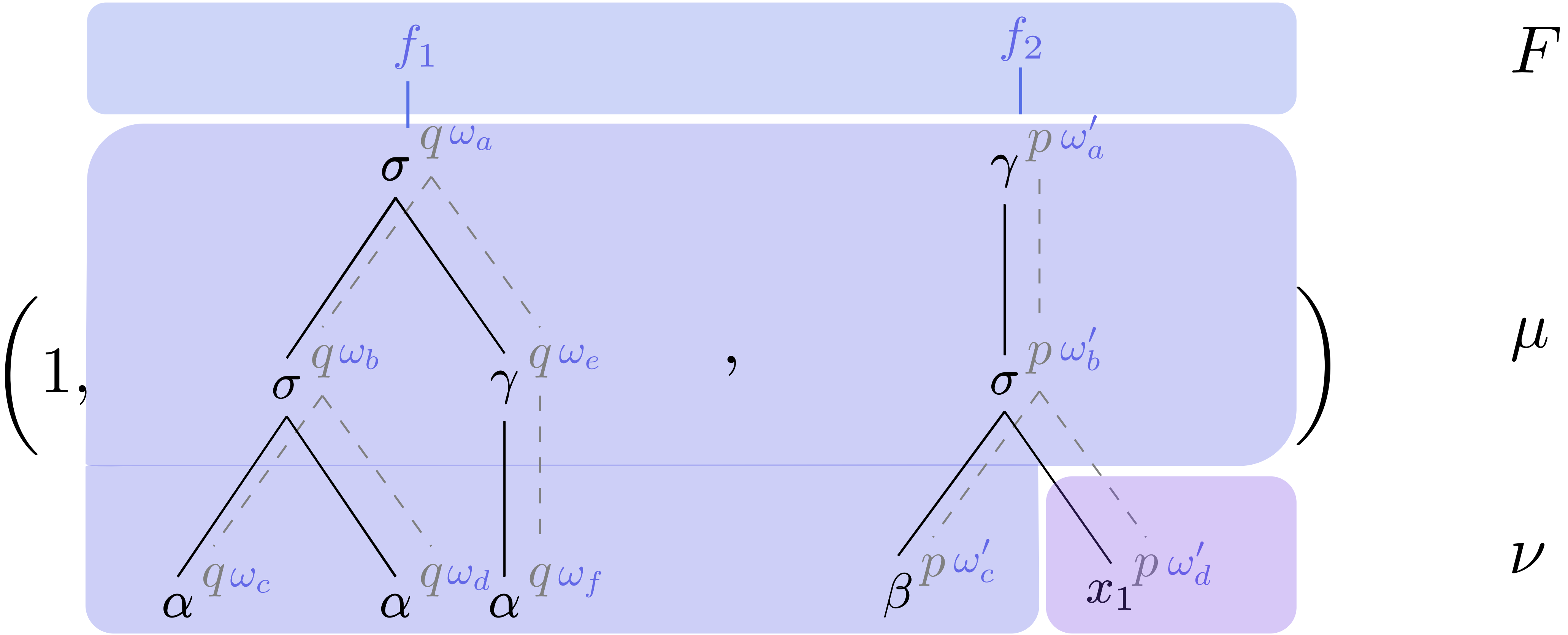

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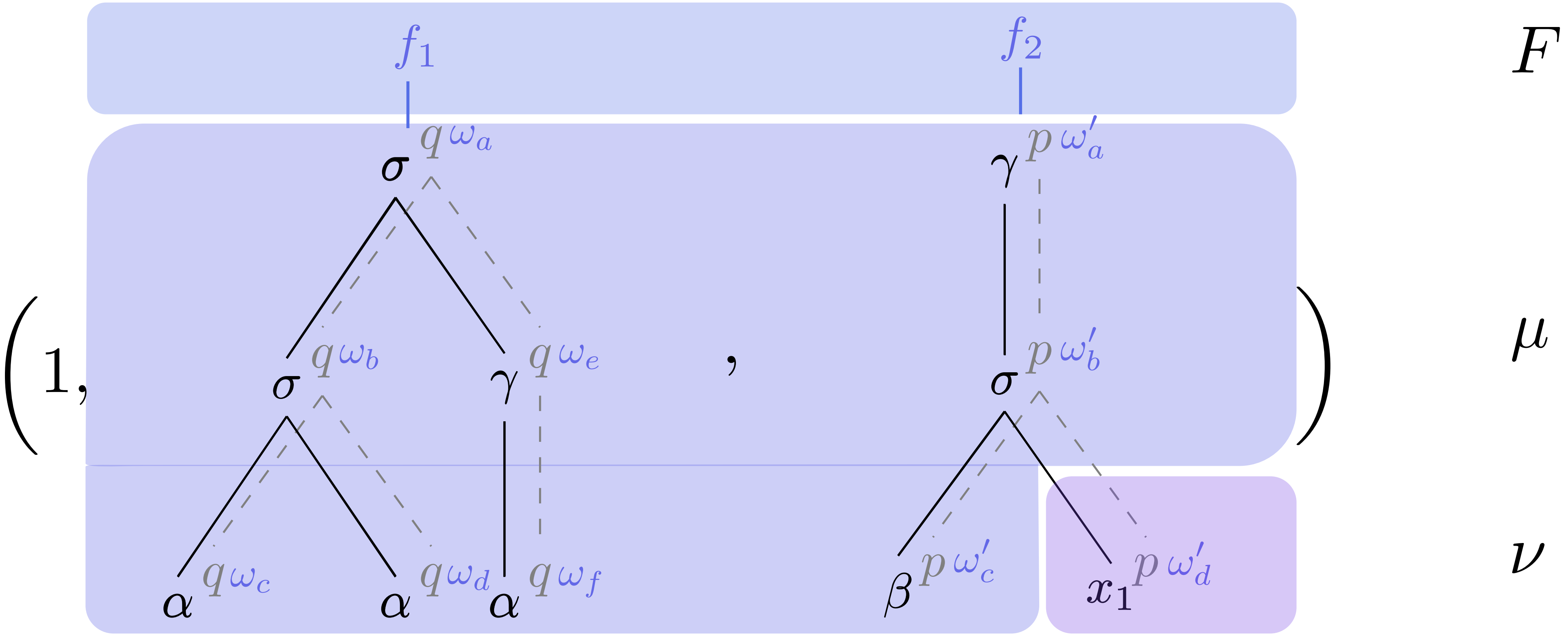

$f_1(\omega_a(\omega_b(\omega_c, \omega_d), \omega_e(\omega_f)))$

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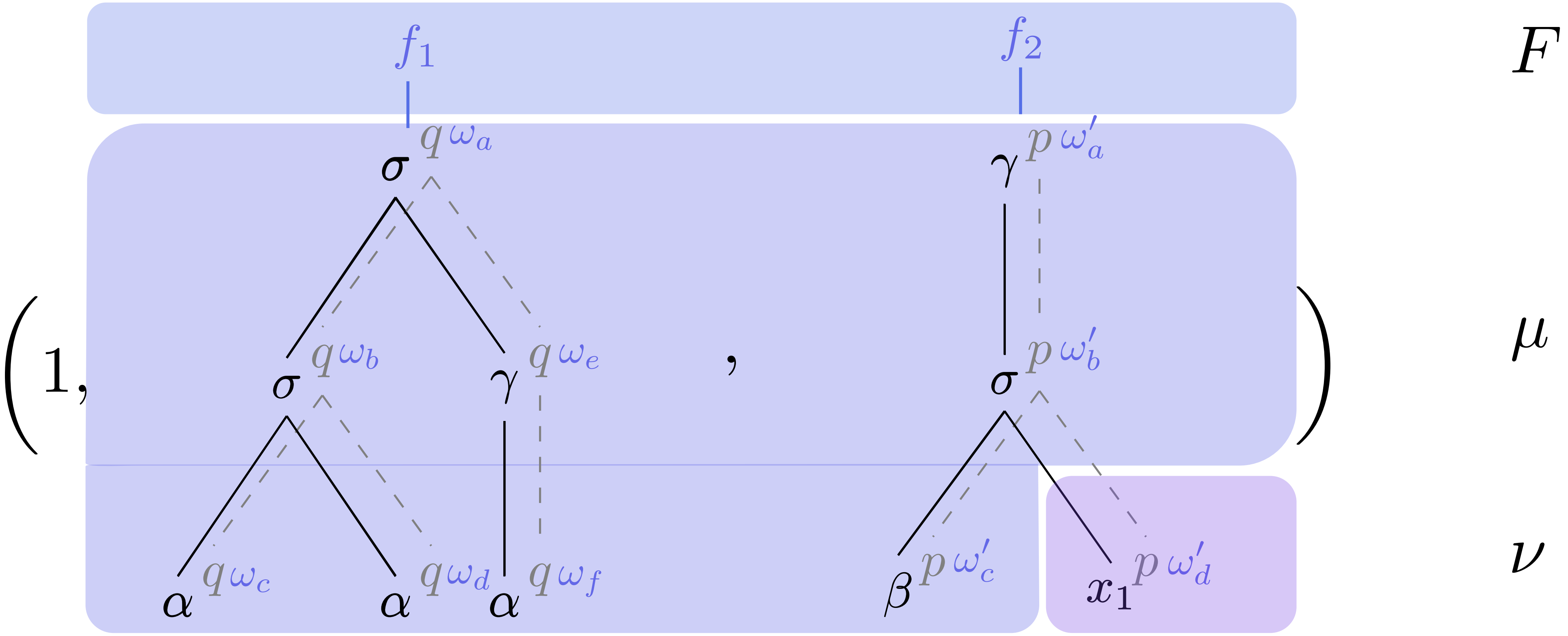

$f_1(\omega_a(\omega_b(\omega_c, \omega_d), \omega_e(\omega_f)))$ $\qquad$ $f_2(\omega'_a(\omega'_b(\omega'_c, \omega'_d)))$

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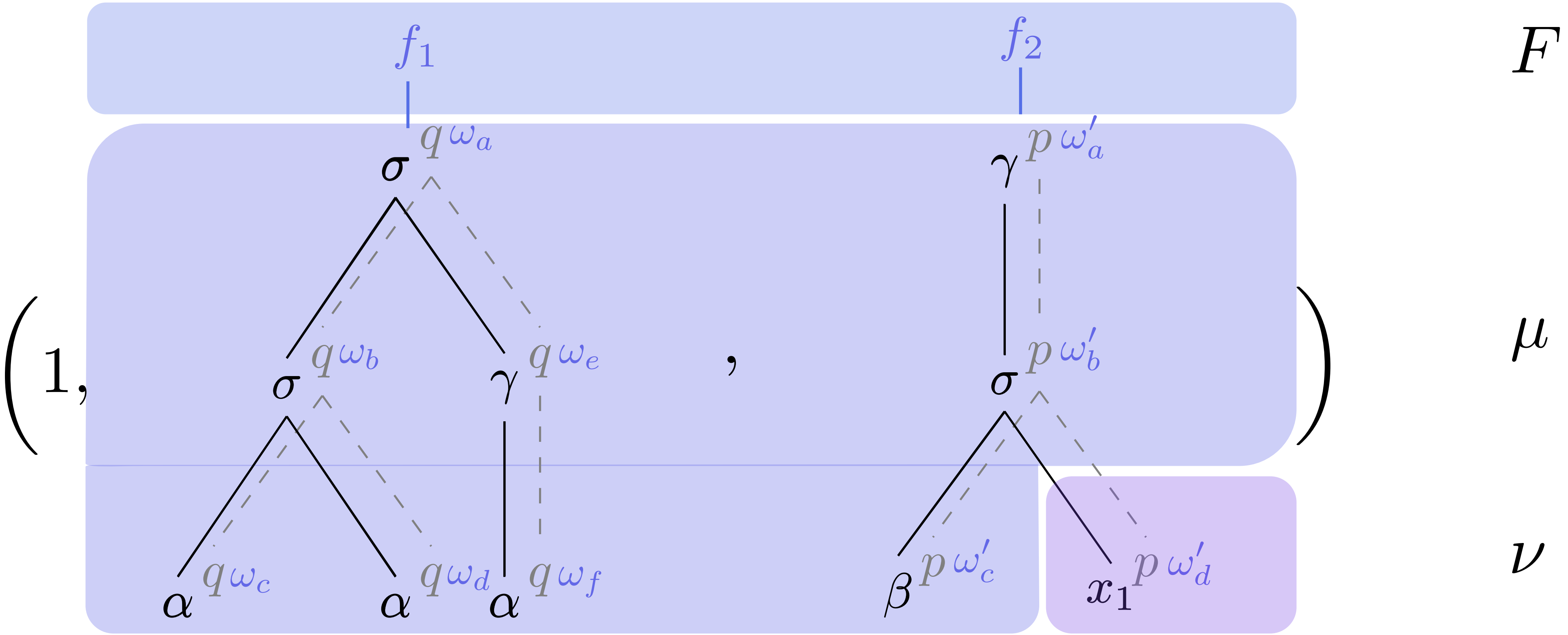

$$\mathrm{wt}(\rho) = \big(\quad f_1(\omega_a(\omega_b(\omega_c, \omega_d), \omega_e(\omega_f))) \quad , \quad f_2(\omega'_a(\omega'_b(\omega'_c, \omega'_d)) \quad \big)$$

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$\exists$ binary operation $\cdot$ on M such that

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How to combine values in $\mathrm{wt}(\rho)$?

M **contains a semiring** if

$\exists$ binary operation $\cdot$ on M such that

$(M, +, \cdot, 0, 1)$ is a semiring and $\forall k\colon \Pi_k \in \Omega_M$

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The **weighted language accepted** by A

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arity = number of variable positions in ξ

4 Rectangularity

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$$[\![A]\!] = \Pi_m([\![A_1]\!], \ldots, [\![A_m]\!])$$

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Proof idea:

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Proof idea:

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$$\vdots$$
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3. Use distributivity

5 Kleene for WTA

From [FulMalVog09]:

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$$\mathrm{Rec}(\Sigma, n, M)$$

accepted by
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From [FulStuVog12]:

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$$\mathrm{MDef}(\Sigma, M)$$

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$\sum_x e$

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From [FulStuVog12]:

$$\mathrm{Rec}_{\mathrm{f}}(\Sigma, 0, M) = \mathrm{MDef}(\Sigma, M)$$

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$\mathrm{H}(\omega) \quad e_1 + e_2$

$\sum_{x} e \quad \sum_{X} e \quad \varphi \rhd e$

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accepted by weighted forest automata with final **states**

definable by forest M-expressions

$$\Pi_m(e_1, \ldots, e_m)$$

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Thank you for your attention!

References

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