The Big-O Problem for Weighted Automata

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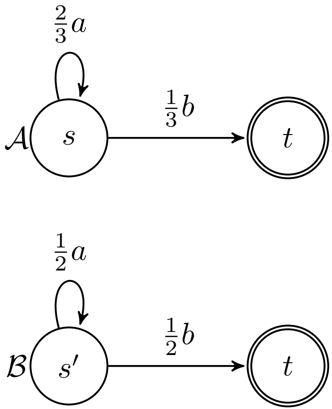
Appeared at CONCUR'20

Model: Non-negative weighted automata

- $(\mathbb{Q}_{\geq 0}, +, \times)$-semi-ring
- LMCs (or HMMs), Probabilistic Automata
- From $\mathcal{A}$ gives weight $\mathcal{A}(w)$ to $w \in \Sigma^*$:
- $\mathcal{A}(a^n\ b) = \left(\frac{2}{3}\right)^n \cdot \frac{1}{3}$

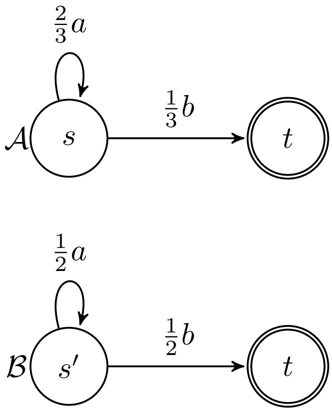


Big-O Problem

Definition (Big-O)

$\mathcal{A}$ is big-O $\mathcal{B}$ if $\exists c > 0$, s.t. $\forall w \in \Sigma^*$

$$\mathcal{A}(w) \leq c \cdot \mathcal{B}(w)$$



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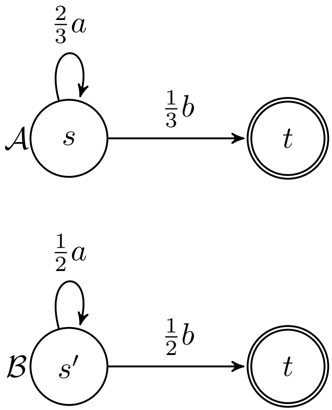
f is big-O g if $\exists c > 0$, s.t. $\forall n \in \mathbb{N}$

$$f(n) \leq c \cdot g(n)$$



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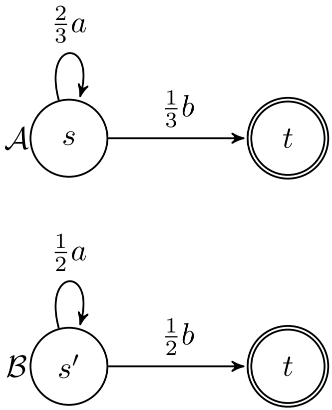

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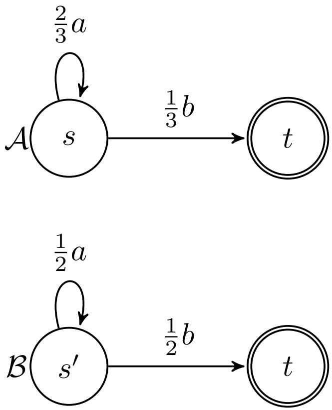

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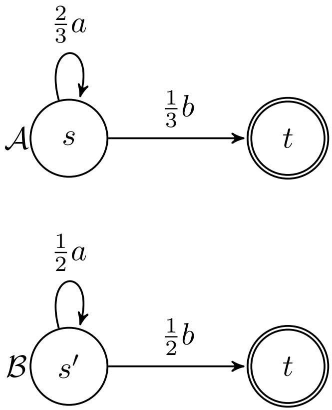
Relaxation of containment



Summary

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$\mathcal{A}$ is big-O of $\mathcal{B}$ if $\exists c > 0$, s.t. $\forall w \in \Sigma^*$

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Results:

- Big-O Problem is undecidable
- On unary weighted automata Big-O Problem is coNP-complete
- The relation to Total Variation Distance
- The relation to Differential Privacy
- Bounded Languages

The language containment condition

$$\begin{array}{rlll} \mathcal{A} \text{ is big-O } \mathcal{B} & \text{if } \exists c > 0, \text{ s.t.} & \forall w \in \Sigma^* & \mathcal{A}(w) \leq c \cdot \mathcal{B}(w) \\ \mathcal{A} \text{ is eventually-big-O } \mathcal{B} & \text{if } \exists c > 0, k > 0, \text{ s.t.} & \forall w \in \Sigma^{\geq k} & \mathcal{A}(w) \leq c \cdot \mathcal{B}(w) \end{array}$$

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Single w with $\mathcal{A}(w) > 0$ and $\mathcal{B}(w) = 0$ then $\mathcal{A}$ is NOT big-O of $\mathcal{B}$.

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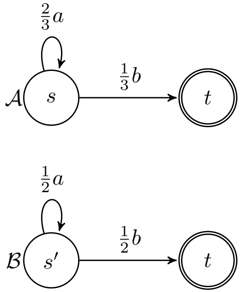
The condition is necessary but not sufficient:

$$L(\mathcal{A}) = L(\mathcal{B}) = \{a^n b \mid n \in \mathbb{N}\}$$



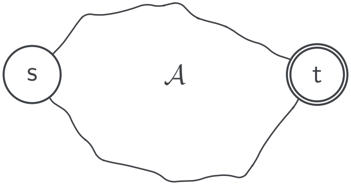
Undecidability

Theorem

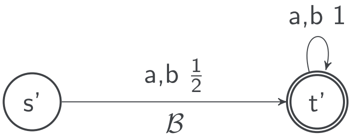
The Big-O Problem is undecidable.

Reduce from Emptiness of Probabilistic Automata ($\exists w \ \mathcal{A}(w) > \frac{1}{2}$)

Reduction from $\exists w\ \mathcal{A}(w) > \frac{1}{2}$?



$$\mathcal{A}(w)$$



$$\mathcal{B}(w) = \frac{1}{2}\ \forall w \in \{a, b\}^+$$

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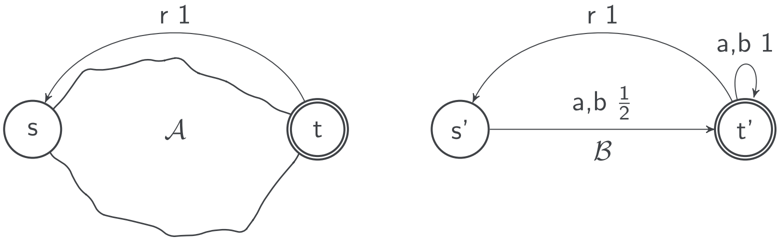

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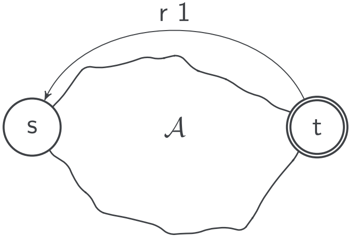

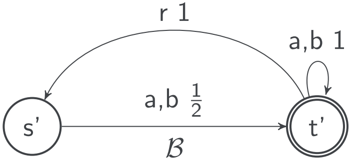

$\mathcal{A}(w)$

$\mathcal{A}(w\,r\,w) = \mathcal{A}(w)\mathcal{A}(w)$

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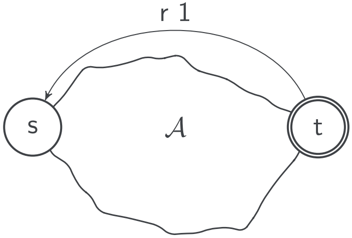

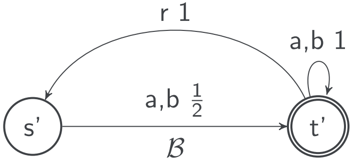

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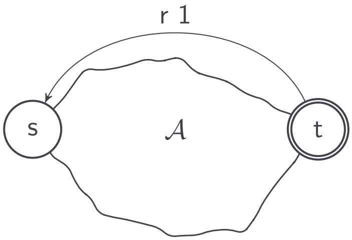
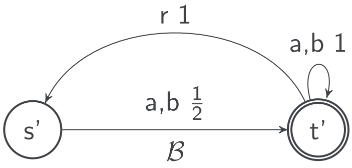
Reduction from $\exists w\ \mathcal{A}(w) > \frac{1}{2}$?





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$$\vdots$$

$$\mathcal{A}((w\,r)^{i-1}w) = (\mathcal{A}(w))^i$$

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$$\frac{\mathcal{A}((w\,r)^{i-1}w)}{\mathcal{B}((w\,r)^{i-1}w)} = (2\mathcal{A}(w))^i$$

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$\mathcal{A}$ is big-O of $\mathcal{B}$ if $\exists c > 0$, s.t. $\forall w \in \Sigma^*$

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Unary Case

$|\Sigma| = 1$.

Theorem

If $\mathcal{A}$ and $\mathcal{B}$ are unary, then the Big-O problem is coNP*-complete*

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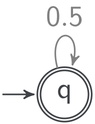
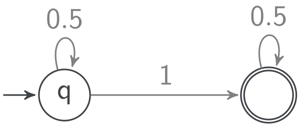
Hardness: Reduce from unary NFA universality.

Unary case



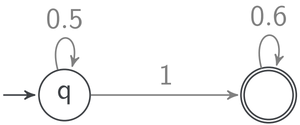
$$\mathcal{A}(a^n) = 0.5^n$$

Unary case



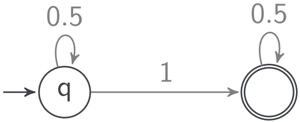
$\mathcal{A}(a^n) \approx 0.5^n n$

Unary case



$\mathcal{A}(a^n) \approx 0.6^n$

Unary case



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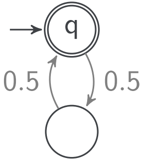
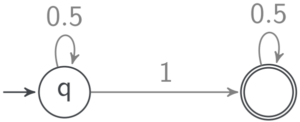


$$\begin{cases} \mathcal{A}(a^n) = 0.5^n & n \text{ even} \\ \mathcal{A}(a^n) = 0 & n \text{ odd} \end{cases}$$

Unary case



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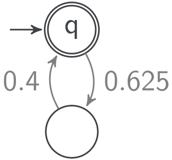

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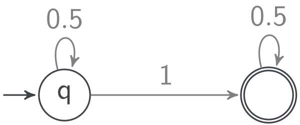

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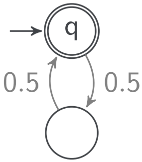

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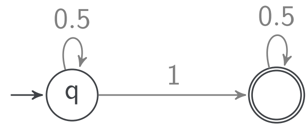

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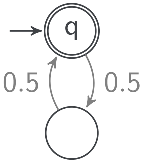

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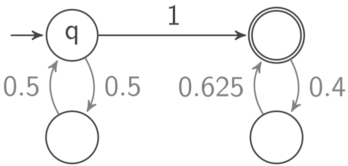

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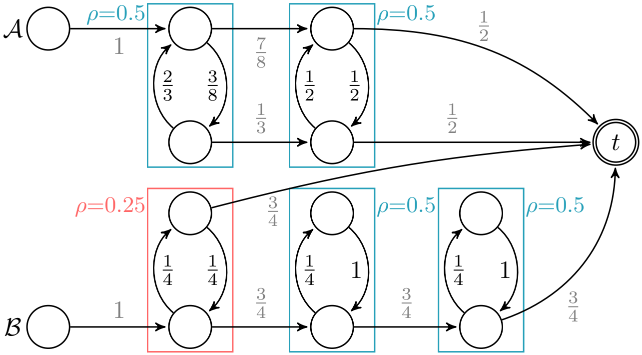
Unary Case Example



$$\mathcal{A}(a^n) = \begin{cases} \Theta(0.5^n n) & n \geq 3 \\ 0 & n = 1, 2 \end{cases}$$

$$\mathcal{B}(a^n) = \begin{cases} \Theta(0.25^n) & n \geq 3, \text{ odd} \\ \Theta(0.5^n n) & n \geq 4, \text{ even} \\ 0 & n = 1, 2 \end{cases}$$

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Then $\mathcal{A}(a^n) \leq c \cdot \mathcal{B}(a^n)$ if and only if

- The language containment condition holds; and
- $f_{\mathcal{A}}(n) \leq f_{\mathcal{B}}(n)$ for all *but finitely many* n.

Unary Case

Verifying $f_{\mathcal{A}}(n) \leq f_{\mathcal{B}}(n)$ for all but finitely many n in coNP:

- for every (ρ, k), for a.b.f.m. n s.t. $f_{\mathcal{A}}(n) = (\rho, k)$ we require $f_{\mathcal{B}}(n) \geq (\rho, k)$

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- verify $L_1 \setminus L_2$ is finite
  - We show this can be done in coNP for two unary NFA.

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$r(\mathcal{A},\mathcal{B})$ is either

- the smallest c such that $\mathcal{A}(w) \leq c\cdot\mathcal{B}(w)$ for every w; or
- ∞ if $\mathcal{A}$ not big-O $\mathcal{B}$

Differential Privacy

Definition

$\mathcal{A}, \mathcal{B}$ are ϵ-differentially private if $\forall w \in \Sigma^*$

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To compute e^{ϵ}, we would like to know if it exists (Big-O Problem)

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$r(\mathcal{A}, \mathcal{B})$, and thus e^ϵ, cannot be approximated

even under the promise that $r(\mathcal{A}, \mathcal{B}) < \infty$

Summary

Definition (Big-O)

$\mathcal{A}$ is big-O of $\mathcal{B}$ if $\exists c > 0$, s.t. $\forall w \in \Sigma^*$

$$\mathcal{A}(w) \leq c \cdot \mathcal{B}(w)$$

Results:

- ✓ Big-O Problem is undecidable
- ✓ On unary weighted automata Big-O Problem is coNP-complete
- ✓ The relation to Total Variation Distance
- ✓ The relation to Differential Privacy
- ▶ Bounded Languages

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A language L is letter-bounded if there exists $a_1, \ldots, a_m \in \Sigma$ such that $L \subseteq a_1^* \ldots a_m^*$

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The big-O problem for weighted automata with bounded languages is decidable subject to Schanuel's conjecture†

†Or the decidability of the first order theory of the *reals* with exponential function

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Open Problems

- When can $r(\mathcal{A}, \mathcal{B})$ (optimal c) be computed/approximated?
- Find the boundary between decidability and undecidability.
  - polynomial-ambiguity is open
- Negative values
- $(\mathbb{N}, \max, +)$ open (we think decidable) and $(\mathbb{N}, \min, +)$ is decidable

2 Letter-bounded Case

Assume $L \subseteq a^*b^*$, then there are two matrices A and B s.t. $\mathcal{A}(a^n b^m) = (A^n \times B^m)_{s,t}$.

Want to know if there exists c such that $(A^n \times B^m)_{s,t} \leq c(A^n \times B^m)_{s',t}$ for all n, m?

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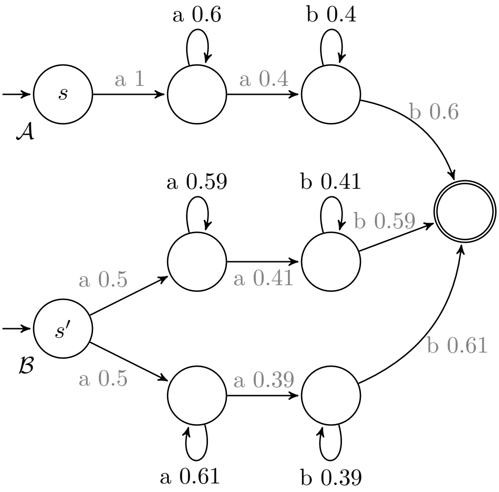
So

- each $(A^n)_{s,q}$ has its own (ρ, k)-value; and
- each $(B^m)_{q,t}$ has its own (ρ, k)-value.

Why is it harder?



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$\mathcal{B}(a^n b^m) = \Theta(0.61^n \cdot 0.39^m + 0.59^n \cdot 0.41^m)$

$$\frac{\mathcal{A}(a^n b^{0.66n})}{\mathcal{B}(a^n b^{0.66n})} \xrightarrow[n \to \infty]{} \infty$$

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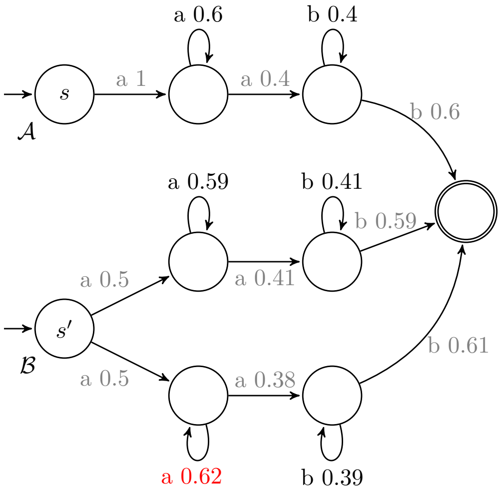

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$$\frac{\mathcal{A}(a^n b^m)}{\mathcal{B}(a^n b^m)} \leq c \text{ for all } n, m$$

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- Encode comparison in to First Order Theory of the *Reals*

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- Encode comparison in to First Order Theory of the *Reals*
  - There is a real solution if and only if there is a integer solution
  - Can relax the real solution to the nearest integer solution, as we are looking at large words
- But encoding $\rho^n n^k$ requires *exponential* function.
  - Decidable subject to Schanuel's conjecture