Evolutionnary dynamics of populations:
Nonlocal PDEs approach

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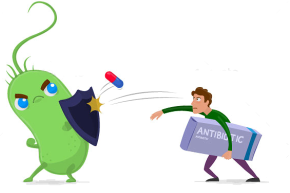
Modeling evolutionary dynamics of population

Main objectives

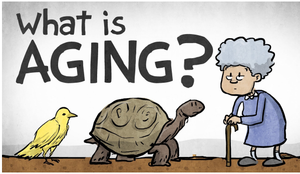
- To **predict** the evolution of organisms (viruses, mammals, trees, bacteria ...), using empirical measures on the model parameters;
- To **understand** interplay between driven forces of evolution – **selection** and **mutation** – in asexuals as well as sexual populations.

→ **Challenges** Adaptation to changing environment



- Better management strategies of resistance emergence. **Antibiotics resistance** is one of the biggest threats to global health, food security and development strategy *(World Health Organization, 2016)*

- Aging might be a source of senescence when environmental conditions vary. *(Cotto and Ronce, 2014)*



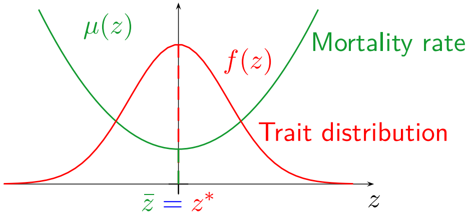
Adaptation and trait distribution

Adaptation: evolutionary process whereby an organism becomes better able to live in its habitat.

Hyp: Adaptation is driven by **mutation** and **selection**.

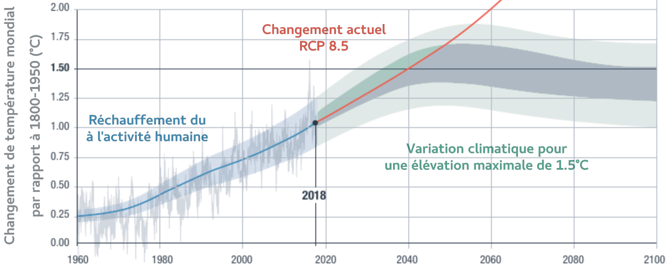
Adaptive trait z quantify the adaptedness of an organism – its survival and reproduction in a given environment – mortality rate $\mu(z)$.

Population density $f(t, z)$ describes the frequency of adaptive trait z inside the population with **mean trait** z^*.



Adaptation occurs when mean trait equal **optimal trait**:
$z^* = \bar{z} = \text{argmin}_{z\in\mathbb{R}}\mu(z)$

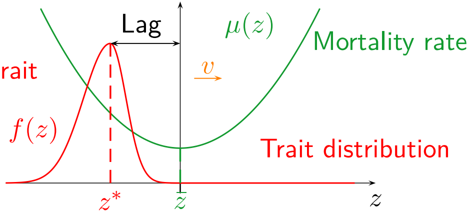
Mal-adaptation and changing environment



(IPCC 2016).

Changing environment modifies **optimal trait** – $\bar{z}(t) := \text{argmin}_{z\in\mathbb{R}}\mu(z,t)$.

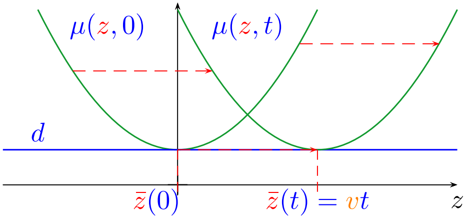
Mal-adaptation occurs when mean trait deviates from **optimal trait**:
$z^* = \bar{z} = \text{argmin}_{z\in\mathbb{R}}\mu(z)$



Asexual and sexual population and changing environment

$$\mu(z,t) := d + \alpha\,(z - vt)^2$$

d: intrinsic mortality rate;
v: speed of change ($v = \bar{z}(t)/t$).



Objectives:

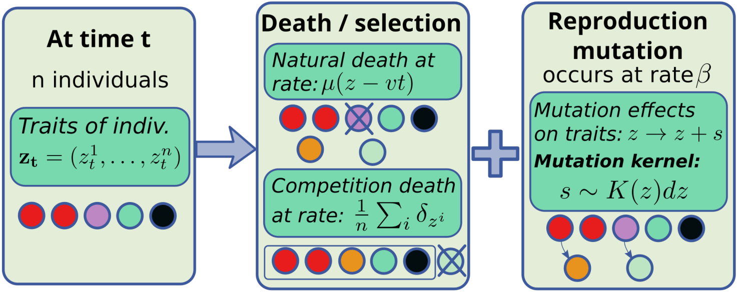
Describe the evolution dynamics under changing environment.

Evolution of asexual population under changing environment

An asexual Individual Based Model

The model describes interplay of mutation, selection and competition.

- **At time** $t = 0$: initial distribution
- Each individual has **3 independent exponential clocks**;



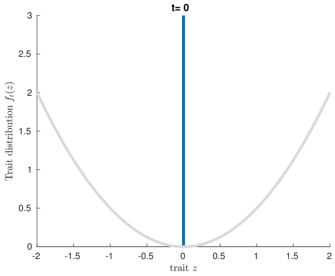
Goal: Describe the trait distribution $\nu_t^n = \dfrac{1}{n} \displaystyle\sum_{i=1}^{N(t)} \delta_{z_t^i}$.

An asexual Individual Based Model

Numerical simulation with $n = 10^3$ individuals

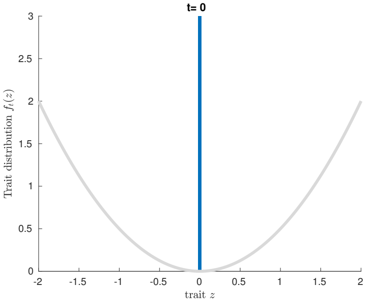
Constant environment $v = 0$

Trait distribution



Gradually changing environment $v > 0$

Trait distribution:



Observation: The trait distribution converge to an equilibrium.

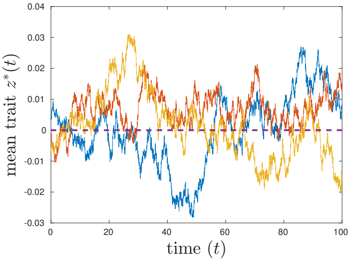
Main objective: Describe the equilibrium of trait distribution.

An asexual Individual Based Model

Numerical simulation with $n = 10^3$ individuals

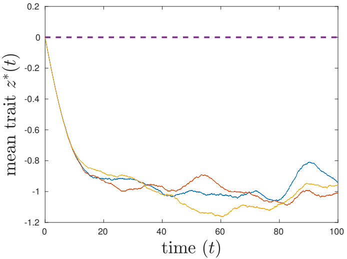
Constant environment $v = 0$

Trait distribution



Gradually changing environment $v > 0$

Trait distribution:



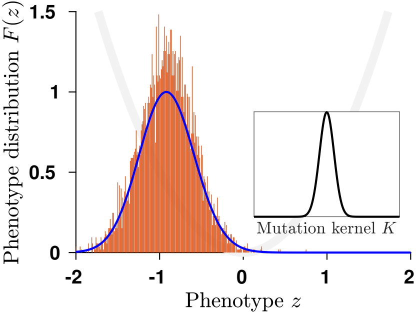
Observation: The mean trait do not always coincide with the optimal trait.

Main objective: Quantify the lag between the mean trait z^* and the optimal trait $\bar{z}$ at the equilibrium.

From stochastic to deterministic model

(Fournier and Méléard, 2004) The sequence of processes ν_t^n converges in law to a deterministic continuous function f_t solution of the following integro-differential equation

$$\begin{aligned}\langle f_t, \phi \rangle &= \langle f_0, \phi \rangle + \int_0^t ds \int_{-\infty}^{\infty} f_s(dz) \beta \int_{-\infty}^{\infty} dz' K(z-z') \phi(z+z') \\ &- \int_0^t ds \int_{-\infty}^{\infty} f_s(dz) \phi(z) \left[\mu(z - vt) + (\beta - d) \int_{-\infty}^{\infty} f_s(dz') \right]\end{aligned}$$



– IBM model with $n = 10^3$
– Deterministic model

Goal: Describe the equilibrium of the deterministic model.

A quantitative genetics model

Population dynamics model: population density $f(t,z)$ described at time (t) with trait (z) by

$$\partial_t f(t,z) + \Big(\mu(z - vt) + (\beta - d)\rho(t)\Big) f(t,z) = \beta \int_{\mathbb{R}} K(z - z') f(t,z')\, dz'$$

Changing environment: optimal trait is moving at speed v, $\bar{z}(t) := vt$.

Mutation kernel: $K(z - z')$ probability that an individuals from trait z' give birth to individuals of trait z.

Selection: Trait z only affects mortality μ,

$$\mu(z) := d + m(z) \quad \text{with} \quad m \ \text{ increasing with } \ |z| \ \text{ and } \ \alpha = m''(0) > 0.$$

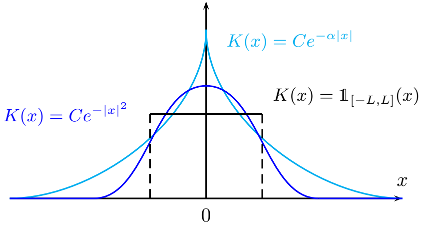
Density–dependence: Mortality increases with the size of the population

$$\rho(t) = \int f(t,z) dz$$

Mutation kernels

Thin-tailed kernel
Rare mutations with large effect

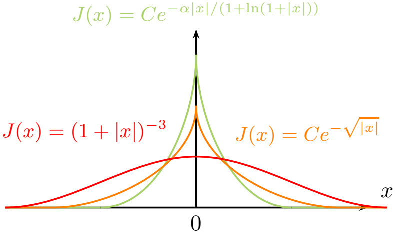
Definition
$$\exists\,\alpha > 0 \text{ such that } \int_{\mathbb{R}} K(x)e^{\alpha x} < \infty.$$



(Diekman et al. 2005, Barles and Perthame 2008)

Fat-tailed kernel
Frequent mutations with large effect

Definition
$$\forall\,\alpha > 0,\ K(x) \geq e^{-\alpha|x|} \quad \text{for large } |x|.$$



(Mirrahimi and Méléard 2014, Bouin et al. 2018)

Selection functions

$$\mu(z) := d + m(z) \quad \text{with} \quad m \quad \text{increasing with} \quad |z| \quad \text{and} \quad \alpha = m''(0) > 0.$$

Super-quadratic functions

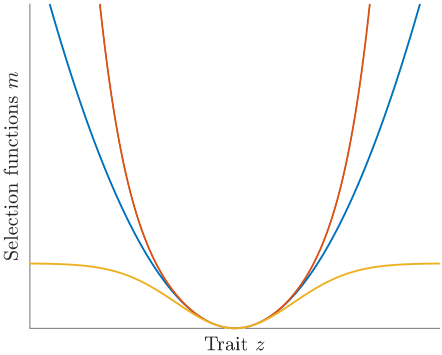
$$m(z) = \alpha z^2/2 + z^6/64$$

Quadratic functions

$$m(z) = \alpha z^2/2$$

Bounded functions

$$m(z) = 1 - e^{-\alpha z^2/2}$$



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Q? Does a mutation/selection equilibrium exists for any speed v?

that is a positive solution F_v of the non-local nonlinear problem in the moving frame $z - vt$

$$-v\partial_z F_v(z) + \mu(z) F_v(z) + (\beta - d) \int_{\mathbb{R}} F_v(z') dz' F_v(z) = \beta \int_{\mathbb{R}} K(z - z') F_v(z')\, dz'$$

that is solution (λ_v, F_v) of the spectral problem

$$-v\partial_z F_v(z) + \mu(z) F_v(z) + \lambda_v F_v(z) = \int_{\mathbb{R}} K(z - z') \beta F_v(z')\, dz' \quad \text{with} \quad \lambda_v > 0.$$

A quantitative genetics model

Population dynamics model: population density $f(t,z)$ described at time (t) with trait (z) by

$$\partial_t f(t,z)+\Big(\mu(z-vt)+(\beta-d)\rho(t)\Big)f(t,z)=\beta\int_{\mathbb{R}}K(z-z')f(t,z')\,dz'$$

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that is solution (λ_v, F_v) of the spectral problem

$$-v\partial_z F_v(z)+\mu(z)F_v(z)+\lambda_v F_v(z)=\int_{\mathbb{R}}K(z-z')\beta F_v(z')\,dz' \text{ with } \lambda_v>0.$$

Q? What is the effect of the speed v? Focus on three main quantities:

(1) **Lag load** $(\Delta\lambda)_v=\lambda_0-\lambda_v$ which measures how v modifies the fitness λ_v.

(2) **Lag** $|z_v^*(t)-\bar{z}(t)|$ which quantifies the distance between the mean trait $z_v^*(t)$ and the optimal trait $\bar{z}(t)$.

(3) **Standing variance** $Var(F_v)$ which quantifies the variability of the population around the mean $z_v^*(t)$

Existence of mutation/selection equilibrium

$$\partial_t f(t,z)+\Big(\mu(z-vt)+(\beta-d)\rho(t)\Big)f(t,z)=\beta\int_{\mathbb{R}} K\,(z-z')f(t,z')\,dz'$$

Mutation/selection equilibrium in constant environment $v=0$.

$$\mu(z)F_0(z)+\lambda_0F_0(z)=\beta\int_{\mathbb{R}} K(z-z')F_0(z')\,dz'$$

Theorem – **Existence of eigenvalue** *(Coville, 2010)*

If the function $(\mu-\inf_{\mathbb{R}}(\mu))^{-1}\notin L^1(\Omega)$ for an open set $\Omega\in\mathbb{R}$, then there exists (λ_0,F_0). Moreover $F_0\in C^0(\mathbb{R})$ and $F_0>0$.

Existence of mutation/selection equilibrium

$$\partial_t f(t,z)+\Big(\mu(z-vt)+(\beta-d)\rho(t)\Big)f(t,z)=\beta\int_{\mathbb{R}}K(z-z')f(t,z')\,dz'$$

Mutation/selection equilibrium for any speed $v>0$?

$$-v\,\partial_z F_v+\mu(z)F_v(z)+\lambda_v F_v(z)=\beta\int_{\mathbb{R}}K(z-z')F_v(z')\,dz'$$

Theorem – **Existence of eigenvalue** *(Coville and Hamel, 2019)*

If the function $\mu\in\mathbb{R}$, and $\mu\geq 0$ in $\mathbb{R}$, then for any speed $v>0$ there exists (λ_v,F_v). Moreover $F_v\in C^1(\mathbb{R})$ and $F_v>0$.

The scaled model with thin–tailed kernels

$$\lambda_v F_v(z) - v\,\partial_z F_v + \mu(z) F_v(z) = \beta \int_{\mathbb{R}} \frac{1}{\sigma} K\left(\frac{z-z'}{\sigma}\right) F_v(z')\,dz' \quad \text{and} \quad \lambda_v > 0.$$

σ^2: mutational variance that generates diversity across generations.
Two trait scales:

Selection scale $Z_{sel} = \left(\frac{\beta}{\alpha}\right)^{1/2}$: measure the strength of selection;

Diversity scale $Z_{div} = \sigma$

Scale ratio ε

$$\varepsilon = \frac{Z_{div}}{Z_{sel}} = \left(\sigma^2 \frac{\alpha}{\beta}\right)^{1/2}$$

The scaled model with thin–tailed kernels

$$\lambda_v F_v(z) - v\,\partial_z F_v + (d + m(z)) F_v(z) = \beta \int_{\mathbb{R}} \frac{1}{\sigma} K\left(\frac{z - z'}{\sigma}\right) F_v(z')\, dz' \quad \text{and} \quad \lambda_v > 0$$

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Scale ratio ε

$$\varepsilon = \frac{Z_{div}}{Z_{sel}} = \left(\sigma^2 \frac{\alpha}{\beta}\right)^{1/2}$$

Rescaled quantitative model with thin–tailed kernels

Rescaled quantitative genetic model: $z \to z/Z_{sel}$

$$\lambda_v F_v(z) - \varepsilon c\, \partial_z F_v + m(z) F_v(z) = \int_{\mathbb{R}} \frac{1}{\varepsilon} K\left(\frac{z-z'}{\varepsilon}\right) F_v(z')\, dz' \text{ and } \lambda_v > \frac{d}{\beta}$$

with $c = \dfrac{v}{\sigma\beta}$, $\lambda_v \to \dfrac{\lambda_v + d}{\beta}$ and $\mu(z) \to m(z)$.

Q? Can we characterize the mutation/selection equilibrium?

!! WITHOUT GAUSSIAN A PRIORI ASSUMPTION !!

Q? What are the effect of the changing speed v?

(1) **Lag load** $(\Delta\lambda)_v = \lambda_0 - \lambda_v$ measuring effects of v on the fitness.

(2) **Lag** $|z_v^* - \bar{z}|$ measuring the adaptation delay?

(3) **Standing variance** $Var(F)$ measuring the variability of the traits around the mean?

Small mutation regime with fat–tailed kernels

$$\lambda_v F_v(z) - v\,\partial_z F_v + \mu(z) F_v(z) = \beta \int_{\mathbb{R}} K\,(z - z') F_v(z')\,dz' \quad \text{and} \quad \lambda_v > 0.$$

Rescaling: We rescale the mutation jumps $z \to z + h$ by $z \to z + \psi_\varepsilon(h)$ with the same probability, where the size of the jump is

$$\psi_\varepsilon(h) = sign(h) \ln(K)^{-1}(\varepsilon K(h))$$

so the rescaled mutation kernel K_ε is given by

$$K_\varepsilon(x) = \frac{1}{\varepsilon} \frac{\ln(K)'(x)}{\ln(K)'(\psi_\varepsilon{}^{-1}(x))} K(x)^\varepsilon \quad \text{and} \quad \int_{\mathbb{R}} (\ln(K_\varepsilon)(x))^2 K_\varepsilon(x) = \varepsilon^2$$

Rescaled model with fat–tailed mutation kernel

$$\lambda_v F_v(z) - \varepsilon c\,\partial_z F_v + m(z) F_v(z) = \int_{\mathbb{R}} K(z') F_v(z - \psi_\varepsilon^{-1}(z'))\,dz' \text{ and } \lambda_v > \frac{d}{\beta}$$

Small mutation regime: Hamilton–Jacobi equation

Thin–tailed kernel

$$\lambda_v F_v(z) - \varepsilon c\,\partial_z F_v + m(z)F_v(z) = \int_{\mathbb{R}} \frac{1}{\varepsilon} K\left(\frac{z-z'}{\varepsilon}\right) F_v(z')\,dz' \text{ and } \lambda_v > \frac{d}{\beta}$$

Thin–tailed kernel

$$\lambda_v F_v(z) - \varepsilon c\,\partial_z F_v + m(z)F_v(z) = \int_{\mathbb{R}} K(z')F_v(z - \psi_\varepsilon^{-1}(z'))\,dz' \text{ and } \lambda_v > \frac{d}{\beta}$$

Distribution transformation

$$F_v^\varepsilon(z) := \exp\left(-\frac{U_v^\varepsilon(z)}{\varepsilon}\right).$$

Main idea when $\varepsilon \to 0$: Taylor expansion according to ε parameter.

$$\begin{aligned} U_v^\varepsilon(z) &= U_v^0(z) + \varepsilon U_v^1(z) + \dots \\ \lambda_v^\varepsilon &= \lambda_v^0 + \varepsilon\lambda_v^1 + \dots \end{aligned}$$

Leading order contribution, lag load and lag

Leading order contribution (λ_v^0, U_v^0) solve the following Hamilton-Jacobi equation

$$\lambda_v^0 + c\partial_z U_v^0(z) + m(z) = 1 + H(\partial_z U_v^0(z)) \text{ and } \lambda_v^0 > \frac{d}{\beta}$$

with the Hamiltonian function H depending on K

$$H(p) = \begin{cases} \int K(y) \exp(yp)\, dy - 1\,. & \text{(thin-tailed)} \\ \int K(y) \exp(\psi(y)p)\, dy - 1\,. & \text{(fat-tailed) } \psi(y) = \dfrac{sign(y)\ln(K)(y)}{\ln(K)'(0)} \end{cases}$$

Selection: $m(z)$ has an optimal trait at $\bar{z} = 0$.

Leading order contribution, lag load and lag

Leading order contribution (λ_v^0, U_v^0) solve the following Hamilton-Jacobi equation

$$\lambda_v^0 + c\partial_z U_v^0(z) + m(z) = 1 + H(\partial_z U_v^0(z)) \text{ and } \lambda_v^0 > \frac{d}{\beta}$$

with the Hamiltonian function H depending on K

Mean fitness: $\lambda_v^0 = 1 - L(c)$ with Lagrangian $L(c) = \sup_{p\in\mathbb{R}} (pc - H(p))$.

- **Selection free** only depends on **mutation**;
- Mean fitness decreases as c increases.
- **Critical speed** c^* such that $\lambda_{c^*} = \frac{d}{\beta}$
  and if $c > c^* \Rightarrow \lambda_c^0 \leq \frac{d}{\beta}$ **EXTINCTION**

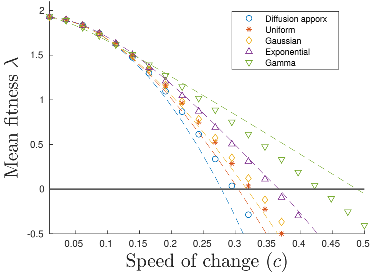
Mean trait: $z_v^* = z_v^{*0}$ roots of $m(z_v^{*0}) = L(c)$

- Lag increases as c increases.

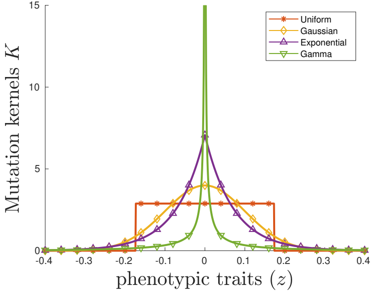
Standing variance: $Var(F) = \varepsilon c / m'(z_0^*)$

- Standing variance depends on c and selection.

Effect of the mutation kernels



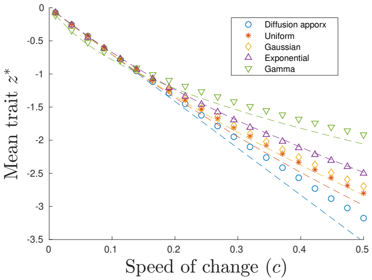
(a) Mean fitness λ



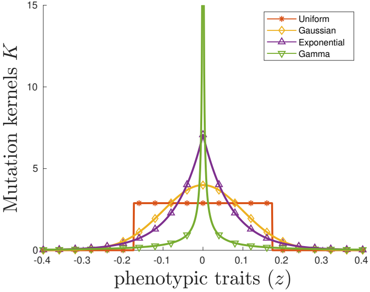
(b) Mutational kernels K

Mean fitness increases with heaviness of the mutation kernels

Effect of the mutation kernels



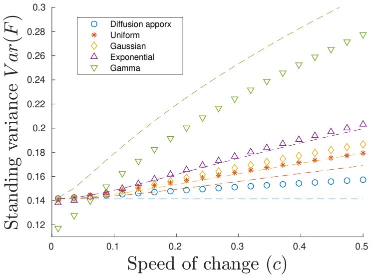
(c) Mean trait z^*



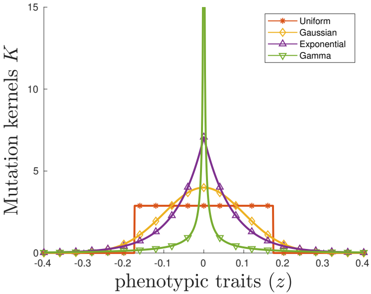
(d) Mutational kernels K

Lag decreases with heaviness of the mutation kernels

Effect of the mutation kernels



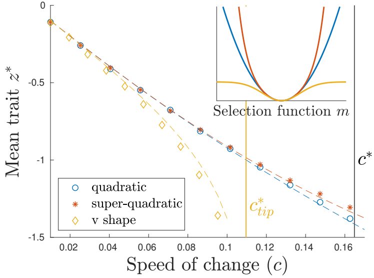
(e) Standing phenotypic variance $Var(F)$



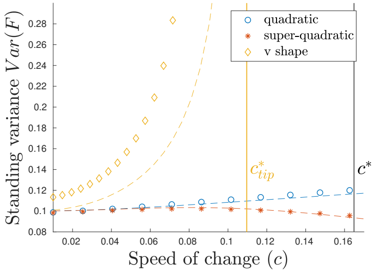
(f) Mutational kernels K

Standing variance increases with heaviness of the mutation kernel

Effect of the selection functions



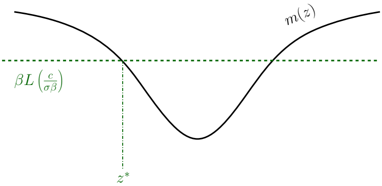
(g) Mean trait



(h) Standing variance

- Lag diverges for bounded selection;

$$c^*_{tip} = \sqrt{2\sigma^2\beta\left(\max_{z\in(-\infty,0)} m(z)\right)}$$

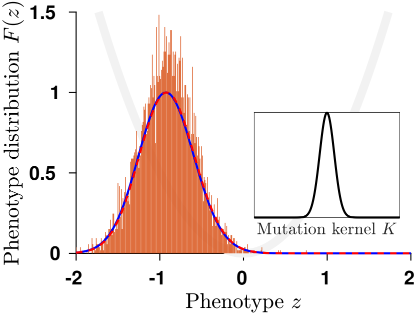


- Standing variance crucially depends on the shape of selection;

Numerical simulations

Mutation kernel: Gaussian kernel $K(z) = exp(-z^2/2\sigma^2)/\sqrt{2\pi\sigma^2}$

Selection: $m(z) = z^2/2$



Scale ratio ε

$\varepsilon = 0.1$ quite large parameter.

Legend

– Approximation formula

- - Deterministic simulation

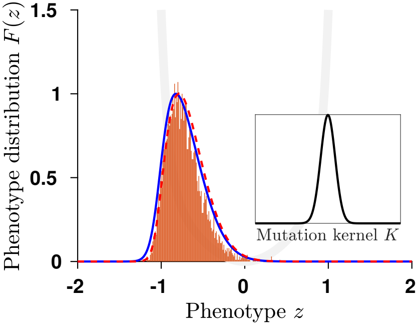
– IBM simulation

Our methodology provides a **good approximation** of the **entire distribution** at equilibrium F_v

Numerical simulations

Mutation kernel: Gaussian kernel $K(z) = exp(-z^2/2\sigma^2)/\sqrt{2\pi\sigma^2}$

Selection: $m(z) = z^2/2 + z^6/64$



Scale ratio ε

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Legend

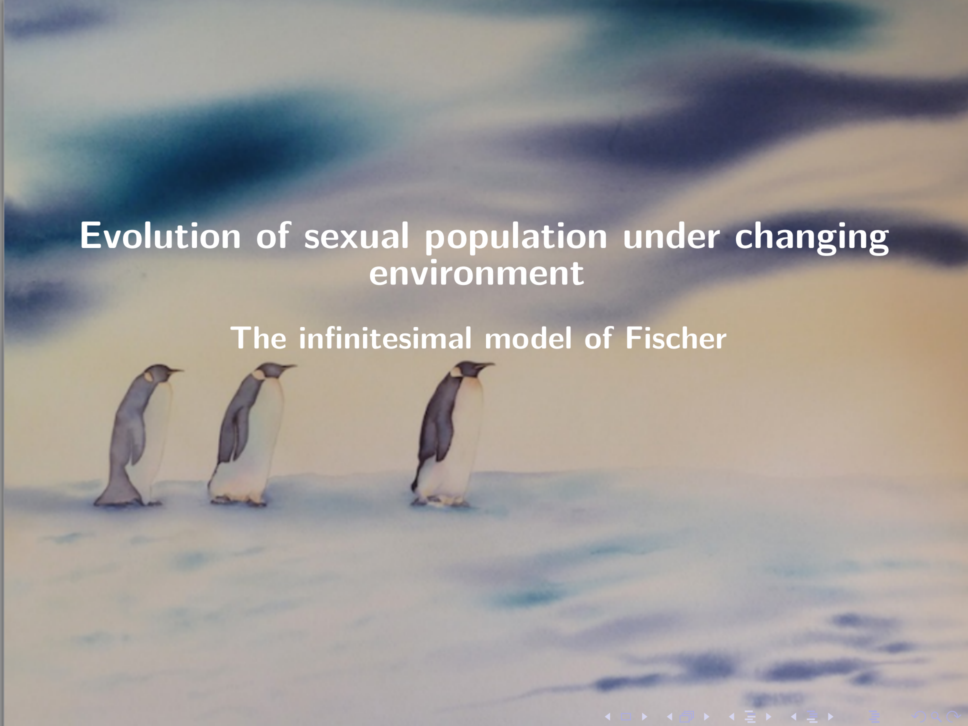
– Approximation formula

- - Deterministic simulation

We are able to track **NON Gaussian distribution**.

Evolution of sexual population under changing environment

The infinitesimal model of Fischer



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$$\partial_t f + \Big(\mu(z - vt) + (\beta - d)\rho(t)\Big) f = \beta \mathcal{B}(f)(t,z)$$

Changing environment: optimal trait is moving at speed v, $\bar{z}(t) := vt$.

Infinitesimal model: offspring phenotype z is drawn randomly around mean of parents traits (z_1, z_2), following Gaussian distribution G^2_σ:

$$\mathcal{B}_\sigma(f)(t,z) = \iint_{\mathbb{R}^2} G_{\sigma^2}\left(z - \frac{z_1 + z_2}{2}\right) f(t,z_1)\left(\frac{f(t,z_2)}{\int_{\mathbb{R}} f(t,z_2')\,dz_2'}\right) dz_1 dz_2$$

Selection: Trait z only affects mortality μ,

$$\mu(z) := d + m(z) \quad \text{with} \quad m \quad \text{increasing with} \quad |z| \quad \text{and} \quad \alpha = m''(0) > 0.$$

Density–dependence: Mortality increases with the size of the population

$$\rho(t) = \int f(t,z)dz$$

A quantitative genetics model

Population dynamics model: population density $f(t,z)$ described at time (t) with trait (z) by

$$\partial_t f(t,z) + \Big(\mu(z - vt) + (\beta - d)\rho(t)\Big) f(t,z) = \beta \mathcal{B}(f)(t,z)$$

Q? Does a mutation/selection equilibrium exists for any speed v?

that is a positive solution F_v of the non-local nonlinear problem in the moving frame $z - vt$

$$-v\partial_z F_v(z) + \mu(z) F_v(z) + (\beta - d)\int_{\mathbb{R}} F_v(z')dz' F_v(z) = \beta \mathcal{B}_\sigma(F_v)(z)$$

that is solution (λ_v, F_v) of the spectral problem

$$-v\partial_z F_v(z) + \mu(z) F_v(z) + \lambda_v F_v(z) = \mathcal{B}_\sigma(F_v)(z) \quad \text{with} \quad \lambda_v > 0.$$

A quantitative genetics model

Population dynamics model: population density $f(t,z)$ described at time (t) with trait (z) by

$$\partial_t f(t,z) + \Big(\mu(z - vt) + (\beta - d)\rho(t)\Big) f(t,z) = \beta \mathcal{B}(f)(t,z)$$

Q? Does a mutation/selection equilibrium exists for any speed v? that is solution (λ_v, F_v) of the spectral problem

$$-v\partial_z F_v(z) + \mu(z) F_v(z) + \lambda_v F_v(z) = \mathcal{B}(F_v)(z) \quad \text{with} \quad \lambda_v > 0.$$

Q? What is the effect of the speed v? Focus on two main quantity:

(1) **Lag load** $(\Delta\lambda)_v = \lambda_0 - \lambda_v$ which measure how v modifies the fitness λ_v.

(2) **Lag** $|z_v^*(t) - \bar{z}(t)|$ which quantifies the distance between the mean trait $z_v^*(t)$ and the optimal trait $\bar{z}(t)$.

(3) **Standing variance** $Var(F_v)$ which quantifies the variability of the population around the mean $z_v^*(t)$.

The scaled model with thin–tailed kernels

$$\lambda_v F_v(z) - v\,\partial_z F_v + \mu(z) F_v(z) = \beta \mathcal{B}_\sigma(F_v)(z)$$

σ^2: variance at linkage equilibrium in absence of selection.
Two trait scales:

Selection scale $Z_{sel} = \left(\frac{\beta}{\alpha}\right)^{1/2}$: measure the strength of selection;

Diversity scale $Z_{div} = \sigma$

Scale ratio ε

$$\varepsilon = \frac{Z_{div}}{Z_{sel}} = \left(\sigma^2 \frac{\alpha}{\beta}\right)^{1/2}$$

The scaled model with thin–tailed kernels

$$\lambda_v F_v(z) - v\,\partial_z F_v + (d + m(z))F_v(z) = \beta \mathcal{B}_\sigma(F_v)(z)$$

σ^2: variance at linkage equilibrium in absence of selection.
Two trait scales:

Selection scale $Z_{sel} = \left(\dfrac{\beta}{\alpha}\right)^{1/2}$: measure the strength of selection;

Diversity scale $Z_{div} = \sigma$

Scale ratio ε

$$\varepsilon = \frac{Z_{div}}{Z_{sel}} = \left(\sigma^2 \frac{\alpha}{\beta}\right)^{1/2}$$

Rescaled quantitative model

Rescaled quantitative genetic model: $z \to z/Z_{sel}$

$$\lambda_v F_v(z) - \varepsilon c\, \partial_z F_v + m(z) F_v(z) = \mathcal{B}_\varepsilon(F_v)(z) \text{ and } \lambda_v > \frac{d}{\beta}$$

with $c = \dfrac{v}{\sigma\beta}$, $\lambda_v \to \dfrac{\lambda_v + d}{\beta}$ and $\mu(z) \to m(z)$.

Q? Can we characterize the mutation/selection equilibrium?

!! WITHOUT GAUSSIAN A PRIORI ASSUMPTION !!

Q? What are the effect of the changing speed v?

(1) **Lag load** $(\Delta\lambda)_v = \lambda_0 - \lambda_v$ measuring effects of v on the fitness.

(2) **Lag** $|z_v^* - \bar{z}|$ measuring the adaptation delay?

(3) **Standing variance** $Var(F)$ measuring the variability of the traits around the mean?

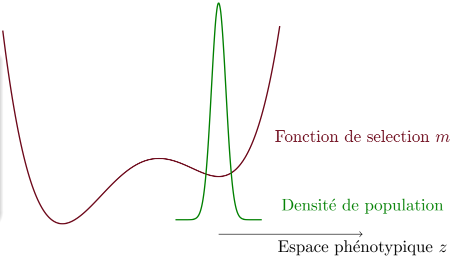
Existence of mutation/selection equilibrium

Mutation/selection equilibrium in constant environment $c = 0$.

$$\lambda_0 F_0(z) + m(z) F_0(z) = \iint_{\mathbb{R}^2} G_{\varepsilon^2}\left(z - \frac{z_1 + z_2}{2}\right) \frac{F_0(z_1)F_0(z_2)}{\int_{\mathbb{R}} F_0(z_2')\, dz_2'}\, dz_1 dz_2 \text{ and } \lambda_0 > \frac{d}{\beta}$$

Theorem – **Existence of equilibrium**
(Calvez, Garnier, Patout, 2019)

For any local minimal z^* of the selection function m and small enough $\varepsilon > 0$, there exists (λ_0, F_0) centered around z^*.



Existence of mutation/selection equilibrium

Mutation/selection equilibrium for any speed $v > 0$?

$$-\varepsilon c\,\partial_z F_v + m(z)F_v(z) + \lambda_v F_v(z) = \beta\mathcal{B}_\varepsilon(F_v)(z)$$

Distribution transformation

$$F_v^\varepsilon(z) := \exp\left(-\frac{U_v^\varepsilon(z)}{\varepsilon^2}\right).$$

Main idea when $\varepsilon \to 0$: Taylor expansion according to ε parameter.

$$\begin{aligned} U_v^\varepsilon(z) &= U_v^0(z) + \varepsilon^2 U_v^1(z) + \dots \\ \lambda_v^\varepsilon &= \lambda_v^0 + \varepsilon^2\lambda_v^1 + \dots \end{aligned}$$

Existence of mutation/selection equilibrium

Mutation/selection equilibrium for any speed $v > 0$?

$$-\varepsilon c\,\partial_z F_v + m(z)F_v(z) + \lambda_v F_v(z) = \beta\mathcal{B}_\varepsilon(F_v)(z)$$

Distribution transformation

$$F_v^\varepsilon(z) := \exp\left(-\frac{U_v^\varepsilon(z)}{\varepsilon^2}\right).$$

Main idea when $\varepsilon \to 0$: Taylor expansion according to ε parameter.

$$\begin{aligned} U_v^\varepsilon(z) &= \frac{1}{2}(z - z_0^*)^2 + \varepsilon^2 U_v^1(z) + \dots \\ \lambda_v^\varepsilon &= \lambda_v^0 + \varepsilon^2 \lambda_v^1 + \dots \end{aligned}$$

Leading order contribution, lag load and lag

Leading order contribution (λ_v^0, U_v^0). It solves the following non–local equation

$$\lambda_v^0 + c\partial_z(z - z_0^*) + m(z) = \exp\left(U_v^1(z_0^*) - 2U_v^1\left(\frac{z + z_0^*}{2}\right) + U_v^1(z)\right) \text{ and } \lambda_v^0 > \frac{d}{\beta}$$

Mean trait: $z_v^* = z_0^*$ roots of $m'(z_0^*) = -c$

- Lag increases as c increases.

Mean fitness: $\lambda_v^0 = 1 - m(z_0^*)$

- Mean fitness decreases as c increases.
- **Critical speed** c^* such that $\lambda_{c^*} = \frac{d}{\beta}$
  and if $c > c^* \Rightarrow \lambda_c^0 \leq \frac{d}{\beta}$ **EXTINCTION**

Standing variance: $Var(F) = \varepsilon^2$

- Standing variance do not depend on c!?

Leading order contribution, lag load and lag

Leading order contribution (λ_v^0, U_v^0)**.** It solves the following non–local equation

$$\lambda_v^0 + c\partial_z(z - z_0^*) + m(z) = \exp\left(U_v^1(z_0^*) - 2U_v^1\left(\frac{z+z_0^*}{2}\right) + U_v^1(z)\right) \text{ and } \lambda_v^0 > \frac{d}{\beta}$$

Mean trait: $z_v^* = z_0^* - \varepsilon^2\left(\frac{m'''(z_0^*)}{2m''(z_0^*)} + 2c\right) + o(\varepsilon^2)$ roots of $m'(z_0^*) = -c$

- Lag increases as c increases.

Mean fitness: $\lambda_v^0 = 1 - m(z_0^*) - \varepsilon^2\left(2c^2 + c\frac{m'''(z_0^*)}{2m''(z_0^*)} + \frac{1}{2}m''(z_0^*)\right) + o(\varepsilon^2)$

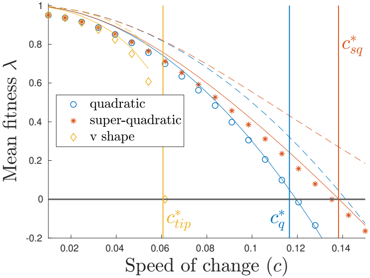
- Mean fitness decreases as c increases.
- **Critical speed** c^* such that $\lambda_{c^*} = \frac{d}{\beta}$

  and if $c > c^* \Rightarrow \lambda_c^0 \leq \frac{d}{\beta}$ **EXTINCTION**

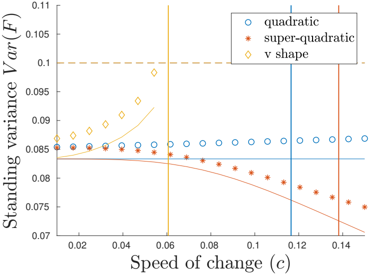
Standing variance: $Var(F) = \dfrac{\varepsilon^2}{1 + 2\varepsilon^2 m''(z_0^*) + o(\varepsilon^2)}$

- Standing variance depends on c and selection.

Effect of the selection functions



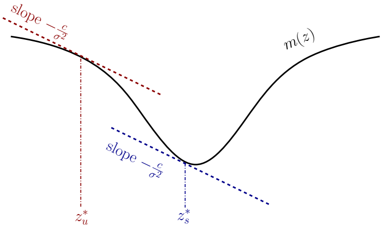
(i) Mean trait



(j) Standing variance

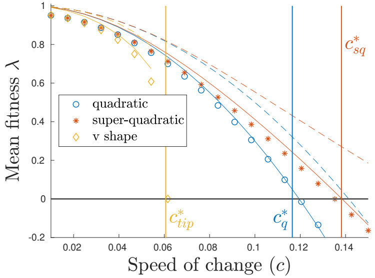
- Lag diverges for bounded selection;

$$c^*_{tip} = \sigma^2 \left(\max_{z \in (-\infty, 0)} |m'(z)| \right)$$

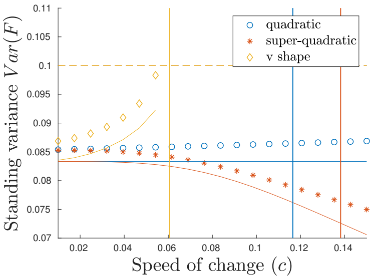


- Standing variance crucially depends on the shape of selection;

Effect of the selection functions



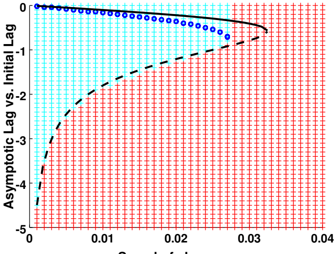
(k) Mean trait



(l) Standing variance

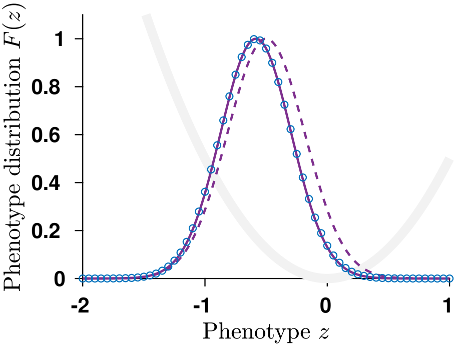
- Lag diverges for bounded selection;

$$c^*_{tip} = \sigma^2 \left(\max_{z \in (-\infty, 0)} |m'(z)| \right)$$



Numerical simulations

Selection: $m(z) = z^2/2$



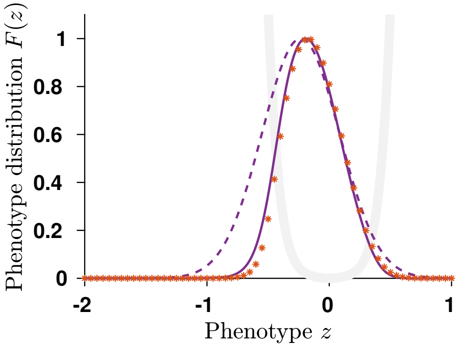
Scale ratio ε
$\varepsilon = 0.1$ quite large parameter.

Legend
- - First order approximation formula
– Second order approximation formula
○ Deterministic simulation

Our methodology provides a **good approximation** of the **entire distribution** at equilibrium F_v

Numerical simulations

Selection: $m(z) = z^2/2 + z^6/64$



Scale ratio ε

$\varepsilon = 0.1$ quite large parameter.

Legend

- - First order approximation formula

– Second order approximation formula

$*$ Deterministic simulation

We are able to track **NON Gaussian distribution**.

Conclusions

$\star$ General methods to describe equilibrium in quantitative genetics models:

- **No Gaussian** a priori on the density distribution F;

- **Description** and **quantification** of the entire distribution, as well as **lag load**, **lag** with **analytical formula**;

- **Flexible methodology** to take into account – general mutation kernel, general selection term, sexual/asexual reproduction;

Thank you for your attention

References

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