POISSON PROCESS APPROXIMATION UNDER STABILIZATION.

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MOTIVATING EXAMPLE

k- NNG : k- nearest neighbour graph.

η_n — Poisson process of intensity 'n' on $\mathbb{T} \cong [0,1]^d$.

$= \{X_1, \ldots, X_{N_n}\}$; $N_n \sim$ Poisson (n) ; X_i i.i.d. $\mathrm{Unif}(\mathbb{T})$.

$$R_k(X_i, \eta_n) = \inf \{ r : \eta_n(B_r(X_i)) \geq k+1 \}.$$

$$\| $$

$$R_k$$

$$\xi[r_n] := \{ (x_i, n|B_{R_k}(x_i)| - a_n) \}_{i \geq 1} \subseteq \mathbb{T} \times \mathbb{R}$$

$$a_n := \log n + (k-1) \log \log n + \log (k-1)!$$

- Various extremal statistics of k-NNG can be deduced from $\xi[r_n]$.
- Extremal edges of k-NNG $\cong$ Extremal edges of MST (M Penrose 1997)
- ξ – gives location of large edges & also length.

THEOREM (M. Penrose 1997)

$\xi[r_n] \xrightarrow{d} \zeta$, Poisson process on $\mathbb{T} \times \mathbb{R}$ with intensity $dx \otimes e^{-t} dt$.

$\xrightarrow{d}$ - weak convergence under vague topology.

THEOREM (M. Otto 2020)

$\xi'[r_n] := \{ X_i : n |B_{R_k}(X_i)| \geq a_n \}$

$D_{TV}(\xi', \zeta') \leq C (\log n)^{-\frac{d-1}{d+1}}$

$\downarrow$ Total variation distance

$$\xi_0[\eta_n] := \{ (x_i, n|B_{R_k}(x_i)| - a_n) : n|B_{R_k}(x_i)| - a_n \geq b_0 \}$$

$$b_0 \in \mathbb{R}$$

THEOREM (Bobrowski, Schulte, Y. 2021)

$$D_{KR}(\xi_0, \zeta_0) \leq C \frac{\log \log n}{\log n}$$

Kantorovich- Rubinstein distance (to be defined)

→ $D_{TV} \leq D_{KR}$. So generalizes Otto `2020.

→ Also gives Penrose 1997.

Other Remarks:

- Penrose (1997) is for homogeneous PPP η_n but Otto (2020) & BSY (2021) apply to η_n with $X_i \stackrel{d}{=} f(x)\,dx$, f cts & $f > 0$.
- Consequence of general approach (will see next)
- Penrose (2018) gives $D_W(\Xi_0(\mathbb{T}\times\mathbb{R}), \zeta(\mathbb{T}\times\mathbb{R}))$.
- Approximation also holds for Binomial process.

GENERAL APPROACH

$\mathbb{X}, \mathbb{Y}$ – LCSCH space ;

μ – finite counting measures in $\mathbb{X}$
(finite multi-sets)

$g(x, \mu) \in \{0, 1\}$, $x \in \mathbb{X}$

$f(x, \mu) \in \mathbb{Y}$

μ_1, μ_2 - finite counting measures / multi-sets.

$$d_{TV}(\mu_1, \mu_2) = \sup_A |\mu_1(A) - \mu_2(A)|$$

$h(\mu)$ is Lipschitz if $|h(\mu_1) - h(\mu_2)| \leq d_{TV}(\mu_1, \mu_2)$

Kantorovich- Rubinstein distance

$$D_{KR}(\xi_1, \xi_2) := \sup_{h \text{ Lip}} |\mathbb{E}[h(\xi_1)] - \mathbb{E}[h(\xi_2)]|$$

ξ_1, ξ_2 - random counting measures / point processes.

$$D_{TV} \leq D_{KR}.$$

η - Poisson process on $\mathbb{X}$ with intensity $\lambda(dx)$.

$$\xi[\eta] = \{ f(x,\eta) : g(x,\eta) = 1 \}$$

LOCAL f, g : $\forall\, x \in \mathbb{X}$, $\exists$ a cpt set $S_x \subseteq \mathbb{X}$

$\ni$ $g(x,\mu) = g(x, \mu \cap S_x)$, $f(x,\mu) = f(x, \mu \cap S_x)$.

L - Intensity measure of ξ.

ζ - Poisson process in $\mathbb{Y}$ with intensity measure $L(dy)$.

THEOREM: (Bobrowski, Schulte, Y. 2021)

$$\xi[\eta] = \{ f(x,\eta) : g(x,\eta) = 1 \}$$

f, g local.

ζ – PPP with same intensity as ξ

$$D_{KR}(\xi, \zeta) \leq 2\left[\mathrm{Var}(\xi(\mathbb{Y})) - \mathbb{E}[\xi(\mathbb{Y})]\right]$$

$$+ 4 \iint \mathbb{1}[S_x \cap S_y \neq \phi]\, \mathbb{E}[g(x, \eta+\delta_x)]\, \mathbb{E}[g(y, \eta+\delta_y)]\, \lambda(dx)\, \lambda(dy).$$

REMARKS:

1. Bounds only involve

Otto 2020

Barbour
& Brown (1992).

Penrose (2018).

→ Generalizes U-statistic bounds $\left[\begin{array}{l} g(\bar{x}, 2) = g(\bar{x}) \\ f(\bar{x}, 2) = f(\bar{x}) \end{array}\right]$
by Decreusefond, Schulte & Thaele
(2016)

(Bobrowski, Schulte, Y. 2021)

f, g local.

ζ – PPP with same intensity as ξ

$$D_{KR}(\xi, \zeta) \leq 2\left[\operatorname{Var}(\xi(\mathbb{Y})) - \mathbb{E}[\xi(\mathbb{Y})]\right] + 4 \iint \mathbb{1}[S_x \cap S_y \neq \phi]\, \mathbb{E}[g(x, \eta+\delta_x)]\, \mathbb{E}[g(y, \eta+\delta_y)]\, \lambda(dx)\, \lambda(dy).$$

GENERATOR APPROACH:

(Generator of spatial birth death process)

[Add-one Cost]

$$\mathbb{E}[h(\zeta)] - h(\mu) = \int_0^\infty \mathcal{L} P_s h(\mu)\, ds.$$

[Decreusefond, Schulte, Thaele '16]