Ergodic Optimization

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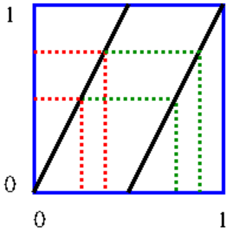
Expanding map example

$T : [0,1] \to [0,1], \quad T(x) = 2x \mod 1.$



Each point has a neighborhood of fixed size where the inverse T^{-1} has $d = 2$ branches, and they are contractions ($\lambda = \frac{1}{2}$-Lipschitz).

Expanding map

X compact metric space.

$T : X \to X$ an expanding map i.e.

$T \in C^0, \quad \exists d \in \mathbb{Z}^+, \quad \exists 0 < \lambda < 1, \quad \exists e_0 > 0 \quad$ s.t.

$\forall x \in X$ the branches of T^{-1} are λ-Lipschitz, i.e.

$\forall x \in X \quad \exists S_i : B(x, e_0) \to X$, $i = 1, \ldots, \ell_x \leqslant d$,

$$d\big(S_i(y), S_i(z)\big) \leqslant \lambda\, d(y, z),$$

$$\begin{cases} T \circ S_i = I_{B(x,e_0)}, \\ S_i \circ T|_{B(S_i(x),\,\lambda e_0)} = I_{B(S_i(x),\,\lambda e_0)}. \end{cases}$$

Main Theorem

X compact metric space.
$T : X \to X$ expanding map, $F \in Lip(X, \mathbb{R})$.
A maximizing measure is a T-invariant Borel probability μ on X such that

$$\int F\, d\mu = \max \left\{ \int F\, d\nu \;\middle|\; \nu \text{ invariant Borel probability} \right\}.$$

Theorem

If X is a compact metric space and $T : X \to X$ is an expanding map then there is an open and dense set $\mathcal{O} \subset Lip(X, \mathbb{R})$ such that for all $F \in \mathcal{O}$ there is a single F-maximizing measure and it is supported on a periodic orbit.

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Maximizing measures are called *ground states* because if $F \geqslant 0$ and μ_β is the invariant measure satisfying

$$\mu_\beta = \underset{\nu \text{ inv. measure}}{\operatorname{argmax}} \left\{ h_\nu(T) + \beta \int F \, d\nu \right\}$$

(the equilibrium state for βF)

($\beta = \frac{1}{\tau}$ = the inverse of the temperature)

then any limit $\lim_{\beta_k \to +\infty} \mu_{\beta_k}$ (a zero temperature limit)
is a maximizing measure (a ground state).

- Bousch, Jenkinson: There is a residual set $\mathcal{U} \subset C^0(X, \mathbb{R})$ s.t. $F \in \mathcal{U} \implies F$ has a unique maximizing measure and it has full support.
- Yuan & Hunt:
  Generically periodic maximizing measures are stable.
  (i.e. same maximizing measures for perturbations of the potential in Hölder or Lipschitz topology.)
  Non-periodic maximizing measures are not stable in Hölder or Lipschitz topology.
- Contreras, Lopes, Thieullen:
  Generically in $C^\alpha(X, \mathbb{R})$ there is a unique maximizing measure.
  If $F \in C^\alpha(X, \mathbb{R})$, then F can be approximated in the C^β topology $\beta < \alpha$ by G with the maximizing measure supported on a periodic orbit.

- Bousch: Proves a similar result for Walters functions:
  $\forall \varepsilon > 0\ \exists \delta > 0$

  $$\forall n \in \mathbb{N}, \quad \forall x, y \in X, \quad d_n(x,y) < \delta \implies |S_nF(x) - S_nF(y)| < \varepsilon.$$

  $$d_n(x,y) := \sup_{i=0,\dots,n} d\big(T^i(x), T^i(y)\big).$$

- Quas & Siefken: prove a similar result for super-continuous functions.
  (functions whose local Lipschitz constant converges to 0 at a given rate: here X is a Cantor set or a shift space).

For example in a subshift of finite type (X is a Cantor set) locally constant functions have periodic maximizing measures.

But those functions are not dense in $C^\alpha(X, \mathbb{R})$ or $Lip(X, \mathbb{R})$. And they are not well adapted for applications to Lagrangian dynamics or twist maps with continuous phase space X.

Write

$$\alpha := \alpha(F) := -\max_{\mu \in \mathcal{M}(T)} \int F \, d\mu. \qquad \text{(Mañé's critical value)}$$

Set of maximizing measures

$$\mathbb{M}(F) := \left\{ \mu \in \mathcal{M}(T) \;\middle|\; \int F \, d\mu = -\alpha(F) \right\}.$$

Generic Uniqueness of maximizing measures

Theorem (Contreras, Lopes, Thieullen)

There is a generic set $\mathcal{G}$ in Lip$(X, \mathbb{R})$ such that

$$\forall F \in \mathcal{G} \qquad \#\mathbb{M}(F) = 1.$$

Moreover, for $F \in \mathcal{G}$, $\mu \in \mathbb{M}(F)$

$$\text{supp}\,\mu \quad \textit{is uniquely ergodic.}$$

Proof:

Enough to prove

$$\mathcal{O}(\varepsilon) := \{ F \in Lip(X, \mathbb{R}) \mid \text{diam}\, \mathbb{M}(F) < \varepsilon\}$$

is open and dense.

Because then take

$$\mathcal{G} := \bigcap_{n \in \mathbb{N}^+} \mathcal{O}(\tfrac{1}{n})$$

will have $\mathcal{G}$ is generic and $\mathcal{G} \subset \{F : \#\mathbb{M}(F) = 1\}$.

Open = upper semicontinuity of $\mathbb{M}(F)$.

(limits of minimizing measures are minimizing)

Density of $\mathcal{O}(\varepsilon)$.

Want to approximate any $F_0 \in Lip(X, \mathbb{R})$ by elements in $\mathcal{O}(\varepsilon)$.

Let $\mathbb{F} = \{f_n\}_{n \in \mathbb{N}^+}$ be a dense set in $Lip(X, \mathbb{R}) \cap [\|f\|_{sup} \leqslant 1]$.
We use $f_0 = -F_0$ the original potential.

$$d(\mu, \nu) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} |\mu(f_n) - \nu(f_n)|.$$

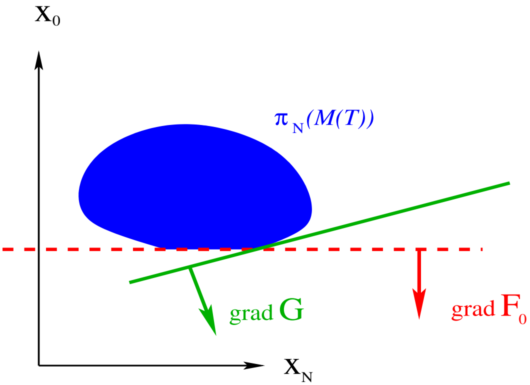
is a metric on $\mathcal{M}(T)$.
Take a finite dimensional approximation of $\mathcal{M}(T)$ by projecting $\pi_N : \mathcal{M}(T) \to \mathbb{R}^{N+1}$ (integrals of test functions)

$$\pi_N(\mu) := \big(-\mu(F_0), \mu(f_1), \ldots, \mu(f_N)\big).$$

$$\text{diam}(\pi_N^{-1}\{x\}) \leqslant \varepsilon_N = \frac{1}{2^N} \to 0.$$

$\alpha(F_0) = \text{argmin}\{\, \mu(-F_0) \mid \mu \in \mathcal{M}(T)\,\}$
$\mathbb{K}_N := \pi_N(\mathcal{M}(T))$ is a convex subset in $\mathbb{R}^{N+1}$
and $[x_0 = \alpha]$ is a supporting hyperplane for $\mathbb{K}_N$.

Use your favourite argument to perturb the hyperplane so that it touches $\mathbb{K}_N$ in a unique (exposed) point $\overline{y}$.



The new supporting hyperplane has normal vector $(1, z_1, \ldots, z_N)$. The touching (exposed) point is

$$\overline{y} := \pi_N(\text{argmin}_{\mathcal{M}(T)}\{-G\}) = \pi_N(\mathbb{M}(G))$$

$$-G = -F_0 + \sum_{n=1}^{N} z_n \cdot f_n$$

Then diam $M(G) \leqslant$ diam $\pi_N^{-1}(\overline{y}) \leqslant \varepsilon_N = \frac{1}{2^N}$.

So $G \in \mathcal{O}(\varepsilon_N)$ and G is very near to F_0. □

Sub-actions = Revelations

A sub-action is a Lispchitz function $u \in Lip(X, \mathbb{R})$ such that

$$F + \alpha \leqslant u \circ T - u.$$

Writing $G := F + \alpha - u \circ T + u$ we have

- G has the same maximizing measures as F.
- $G \leqslant 0$.
- For $\mu \in \mathcal{M}_{max}(F)$ we have $\int G \, d\mu = 0$.

$$\therefore \quad \mu \in \mathbb{M}(F) \quad \Longleftrightarrow \quad \text{supp}(\mu) \subset [G = 0].$$

If u exists: On the support of a maximizing measure $G = 0$, i.e. $F + c$ is a coboundary.

Generic unique ergodicity

If we construct a sub-action

$$\mu \in \mathbb{M}(F) \iff \operatorname{supp}(\mu) \subset [G = 0]$$

If $\mathbb{M}(F) = \{\mu\}$ then

μ is the unique invariant measure in $\operatorname{supp}(\mu)$.

One can construct sub-actions as “maximal profits” or “optimal values” along pre-orbits. For example

$$u(x) = \sup \left\{ \sum_{k=0}^{n-1} \{ F(T^k y) - \alpha \} \;\middle|\; T^n(y) = x, \; y \in X \right\}$$

will be a sub-action.

Also

- Defining a “Mañé action potential”.
- Using methods from “Weak KAM Theory”.

Lax Operator

$$\mathcal{M}(T) := \{\ T\text{-invariant Borel probabilities}\ \}$$

$F \in Lip(X, \mathbb{R})$, $\qquad \mathcal{L}_F : Lip(X, \mathbb{R}) \to Lip(X, \mathbb{R})$:

$$\mathcal{L}_F(u)(x) := \max_{y \in T^{-1}(x)} \{\, \alpha + F(x) + u(x)\, \},$$

$$\text{where} \quad \alpha := -\max_{\mu \in \mathcal{M}(T)} \int F\, d\mu.$$

Set of maximizing measures

$$\mathbb{M}(F) := \Big\{ \mu \in \mathcal{M}(T) \;\Big|\; \int F\, d\mu = -\alpha(F) \Big\}.$$

Calibrated sub-action = Fixed point of Lax Operator
= Solution to Bellman equation.

$$\mathcal{L}_F(u) = u$$

write

$$\overline{F} := F + \alpha + u - u \circ T.$$

REMARKS:

1. $$-\alpha(\overline{F}) = \max_{\mu \in \mathcal{M}(T)} \int \overline{F}\, d\mu = 0.$$
2. $\overline{F} \leqslant 0.$
3. $$\mathbb{M}(\overline{F}) = \mathbb{M}(F) = \left\{ \mu \in \mathcal{M}(T) \;\middle|\; \operatorname{supp}(\mu) \subset [\, \overline{F} = 0 \,] \right\} .$$

Proposition

If F is Lipschitz then
there exists a Lipschitz calibrated sub-action.

Proof.

1. Prove that $\mathrm{Lip}\,(\mathcal{L}_F(u)) \leqslant \lambda\,\big(\mathrm{Lip}(u) + \mathrm{Lip}(F)\big)$.
2. Then $\mathcal{L}_F$ leaves invariant the space

$$\mathbb{E} := \Big\{\, u \in Lip(X,\mathbb{R}) \;\Big|\; \mathrm{Lip}(u) \leqslant \frac{\lambda\,\mathrm{Lip}(F)}{1-\lambda} \,\Big\}.$$

3. $\mathbb{E}/_{\{\text{constants}\}}$ is compact & convex.
   $\mathcal{L}_F$ is continuous on $\mathbb{E}$.
   Schauder Thm. $\Longrightarrow$ $\mathcal{L}_F$ has a fixed pt. on $\mathbb{E}/_{\{\text{constants}\}}$.
4. Prove it is a fixed point on $\mathbb{E}$.

□

1. If u is a calibrated sub-action:
Every point $z \in X$ has a calibrating pre-orbit $(z_k)_{k \leqslant 0}$ s.t.

$$\begin{cases} T^i(z_{-i}) = z_0 = z, & \forall i \geqslant 0; \\ u(z_{k+1}) = u(z_k) + \alpha + F(z_k), & \forall k \leqslant -1. \end{cases}$$

Equivalently, since $T(z_k) = z_{k+1}$,

$$\overline{F}(z_k) = 0 \qquad \forall k \leqslant -1.$$

Proposition

If $\mathcal{O}(y) \subset X$ is a periodic orbit such that for every calibrated sub-action the α-limit of any calibrating pre-orbit is in $\mathcal{O}(y)$ then every maximizing measure has support in $\mathcal{O}(y)$. →3

Need:

$$\forall \mu \in \mathbb{M}(F) \quad \mathit{supp}(\mu) \subset \alpha\text{-limit of calibrating pre-orbits}$$
$$\subset \alpha\text{-limit of orbits in } [\overline{F} = 0]$$
$$\textit{enough}: \quad \subset \alpha\text{-limit of orbits in } \operatorname{supp}(\mu).$$

For example:

Extend T to an invertible dynamical system $\mathbb{T} : \mathbb{X} \to \mathbb{X}$. Lift μ to a $\mathbb{T}$-invariant $\overline{\mu}$. The set $\mathbb{Y}$ of recurrent points of $\mathbb{T}^{-1}$ in $\operatorname{supp}(\overline{\mu})$ has total $\overline{\mu}$-measure and projects onto a set $Y = \pi(\mathbb{Y})$ with total measure of points which are α-limits of pre-orbits in $\operatorname{supp}(\mu)$.

Shadowing Lemma

Definition

1. $(x_n)_{n\in\mathbb{N}} \subset X$ is a δ-pseudo-orbit if
$d\big(x_{n+1}, T(x_n)\big) \leqslant \delta, \quad \forall n \in \mathbb{N}$.
2. A point $y \in X$ ε-shadows a pseudo-orbit $(x_n)_{n\in\mathbb{N}}$ if
$d\big(T^n(y), x_n\big) < \varepsilon, \quad \forall n \in \mathbb{N}$.

Proposition (Shadowing Lemma)

If $(x_k)_{k\in\mathbb{N}}$ is a δ-pseudo-orbit
$\Longrightarrow$ $\exists y \in X$ *whose orbit* ε*-shadows* (x_k)
with $\varepsilon = \frac{\delta}{1-\lambda}$.

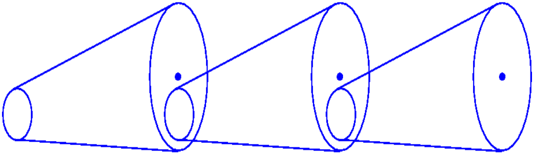
If (x_k) is periodic
$\Longrightarrow$ *y is a periodic point with the same period.*

Proof.

$a = \frac{\lambda\delta}{1-\lambda}$.

$\{y\} = \bigcap_{k=0}^{\infty} S_0 \circ \cdots \circ S_k\big(B(x_{k+1}, a)\big)$.

where the inverse branch S_k is chosen such that $S_k(T(x_k)) = x_k$. □



2 survey articles in Ergodic Optimization by

Oliver Jenkinson.

survey in optimization of Lyapunov exponents by

Djairo Bochi

Zero entropy

Theorem (Morris)

Let X be a compact metric space and $T : X \to X$ an expanding map. There is a residual set $\mathcal{G} \subset Lip(X, \mathbb{R})$ such that if $F \in \mathcal{G}$ then there is a unique F-maximizing measure and it has zero metric entropy.

Idea of the Proof.

1. *Use estimates of Bressaud & Quas to obtain a close return in* $\mathrm{supp}(\mu)$ *which is not too long in time.*
   Construct a periodic orbit L_n *with it. It has an action proportional to the distance of the return.*
2. *Use* $f_n(x) := f(x) - \varepsilon\, d(x, L_n)$
   If a measure ν *is nearby the closed orbit* L_n*, then it has small entropy.*
   If it is far from L_n *then it is not minimizing for the perturbed function* f_n*.*
   Those f_n *form a dense set.*

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Lemma

Σ_A sub-shift of finite type with M symbols and entropy h.
Then
Σ_A contains an orbit of period at most $1 + M\, e^{1-h}$.

Proof:

$k+1 =$ period of shortest periodic orbit in Σ_A.

Claim

A word of length k is determined by the symbols it contains.

No periodic orbits of period $\leqslant k \Longrightarrow$ any allowed k-word contains k distinct symbols. (a repeated symbol gives a shortest closed orbit).

Suppose by contradiction

$\exists\ u, v$ distinct words of length k with the same symbols.

$\Longrightarrow \exists$ symbols a, b such that

consecutive $(ab) \in u$ and inverse order$(b \cdots a) \in v$ (length $\leqslant k$)

then $(b \cdots a)(b \cdots a)(b \cdots a) \cdots$ is an allowed periodic orbit in Σ_A of period $\leqslant k$ $(\Rightarrow\Leftarrow)$

All k-words have distinct symbols, $M = \#\text{alphabet}$

$\Longrightarrow$ at most $\binom{M}{k}$ words of length k.

$\ell \longmapsto \#$words of period $\ell =: W(\ell)$ is sub-multiplicative in Σ_A

(not all can concatenate)

$$\Longrightarrow \qquad h_{top}(\Sigma_A) = \text{exp growth of periodic orbits}$$

$$\leqslant \inf_{\ell} \tfrac{1}{\ell} \log W(\ell) \leqslant \frac{1}{k} \log \binom{M}{k}.$$

$$e^{h_{top}\, k} = e^{hk} \leqslant \binom{M}{k} \leqslant \frac{M^k}{k!} \leqslant \left(\frac{M\, e}{k}\right)^k$$

Taking k-root

$$k \leqslant M\, e^{1-h}.$$

$$\text{minimal period } = k + 1 \leqslant 1 + M\, e^{1-h}. \quad \square$$

Bressaud & Quas where interested in how well a maximizing measure could be approximated by periodic orbits.

Let $\mu \in \mathbb{M}(T)$ be a maximizing measure and $K := \text{supp}(\mu)$.

$$c(\nu, K) := \sup_{x \in \text{supp}(\nu)} d(x, K).$$

Proposition (Bressaud & Quas (2007))

$$\forall k \in \mathbb{N} \qquad \lim_{n \to \infty} n^k \left(\inf_{\nu \in P_n(T)} c(\nu, K) \right) = 0$$

Sketch of proof

Let $N > 0$, $\quad 0 < \delta <$ expansivity constant for K.

$G =$ minimal (N, δ)-generating set for K.

Let $\Sigma_A \subset G^{\mathbb{N}}$ be the sub-shift of finite type with symbols in G and matrix $A \in \{0,1\}^{G \times G}$ defined by

$$A(x,y) = 1 \quad \iff \quad \sup_{0 \leqslant k < N} d(T^k(T^N x), T^k y) < \delta.$$

Apply the Lemma to the shift Σ_A (get small periodic orbit).

Use the shadowing lemma to define $\pi : \Sigma_A \to$ neighbourhood of K. (N large $\Longrightarrow$ smaller neighbourhood)

Finish the estimates.

Corollary

There are sequences of integers $m_n \in \mathbb{N}^+$ *and periodic orbits* $\mu_n \in P_{N_n}(T)$ *with period* N_n *such that*

$$\forall \beta \in]0,1[$$

$$\int d(x,K)\, d\mu_n(x) = o(\beta^{m_n}) \quad \text{and} \quad \lim_n \frac{\log N_n}{m_n} = 0.$$

$$K = \operatorname{supp} \mu, \qquad \mu \in \mathbb{M}(T).$$

Just algebraic manipulations from Bressaud & Quas Proposition.

Zero Entropy: Part II.

$\mathbb{M}(F)$ = maximizing measures for F

$$\mathcal{E}(\gamma) := \{\, F \in Lip(X,\mathbb{R}) \mid h(\mu) < 2\,\gamma\, h_{top}(T) \quad \forall \mu \in \mathbb{M}(F) \,\}$$

$$\mathcal{O} := \{\, F \in Lip(X,\mathbb{R}) \mid \#\mathbb{M}(F) = 1 \,\}$$

Enough to prove $\forall \gamma > 0 \quad \mathcal{E}(\gamma)$ is open and dense.
Because then

$$\mathcal{G} := \mathcal{O} \cap \bigcap_{n\in\mathbb{N}} \mathcal{E}(\tfrac{1}{n})$$

is the required generic set.

1. $\mathcal{E}(\gamma)$ is open:
   (prove that the complement is closed using the semicontinuity of $\mathbb{M}$ and the semicontinuity of the entropy)
2. $\mathcal{E}(\gamma)$ is dense.

Let N_n (= period), m_n, μ_n from the Corollary,

$$L_n := \text{supp}(\mu_n)$$ (periodic orbit very near K and small period)

Lemma

There are $0 < \theta = \theta(T) < 1$ *and* $K_\gamma > 0$ *such that if* $n > K_\gamma$, $\nu \in \mathcal{M}(T)$, $h(\nu) \geqslant 2\gamma\, h_{top}(T)$
Then

$$\nu(\{\, x \in X \mid d(x, L_n) \geqslant \theta^{m_n} \}) > \gamma.$$

Lemma $\Longrightarrow$ density of $\mathcal{E}(\gamma)$

Replacing F by $\overline{F} = F + \alpha(F) + u \circ T - u$ can assume that $F \leqslant 0 = \max F$, F is Lipschitz. Then

$$F(x) \leqslant C\, d(x, K), \qquad K = \operatorname{supp}(\mu)$$

Perturb the potential F by

$$F_n(x) := F(x) - \beta\, d(x, L_n).$$

Want to show $F_n \in \mathcal{O}(\gamma)$ i.e. $\forall \nu \in \mathbb{M}(F_n) \quad h(\nu) \leqslant 2\gamma\, h_{top}(T)$.

If $h(\nu) > 2\gamma\, h_{top}(T)$ use the estimate of the Lemma

$$\int d(x, L_n)\, d\nu = \theta^{m_n}\ \nu([x : d(x, L_n) \geqslant \theta^{m_n}]) \geqslant \gamma\, \theta^{m_n}$$

And $F \leqslant Cd(x, K)$ to show that in this case ν can not be maximizing for F_n.

Since $\int d(x, L_n)\, d\nu \geqslant \gamma\, \theta^{m_n}$
and by the Corollary 1 $\int d(x, K)\, d\mu_n(x) = o(\theta^{m_n})$
we can choose n such that

$$\beta \int d(x, L_n)\, d\nu > C \int d(x, K)\, d\mu_n$$

$$\begin{aligned}\int F_n\, d\nu &= \int \overline{F}\, d\nu - \beta \int d(x, L_n)\, d\nu \\ &< 0 - C \int d(x, K)\, d\mu_n \\ &\leqslant \int \overline{F}\, d\mu_n = \int \overline{F}_n\, d\mu_n \leqslant -\alpha(F_n).\end{aligned}$$

$\Longrightarrow \nu$ is not F_n-maximizing. □

Proof of the Lemma

Use a Markov partition $\mathbb{P}$ for T of small diameter $<$ expansivity ctant.

$h(\nu) = h(\nu, \mathbb{P}) \leqslant \frac{1}{m_n} H(\nu, \mathbb{P}^{(m_n)})$ $\qquad \mathbb{P}^{(m_n)} = \bigvee_{k=0}^{m_n-1} T^{-k}\mathbb{P}$

$W_n := \{A \in \mathbb{P}^{(m_n)} : d(x; L_n) < \theta^{m_n}$ for some $x \in A\}$

Estimate entropy by

$$h(\nu) \leqslant \frac{1}{m_n} \sum_{A \in W_n} \nu(A) \log \nu(A) + \frac{1}{m_n} \sum_{A \notin W_n} \nu(A) \log \nu(A)$$

very small entropy near the periodic orbit $\qquad$ entropy must come from W_n^c

Then estimate $h(\nu) > 2\gamma\, h_{top}(T) \Longrightarrow \nu(\bigcup W_n^c) > \gamma$.

- Original argument: Yuan & Hunt.
- Present argument: Quas & Siefken.
- Adapted to pseudo-orbits.

$$Per(T) := \bigcup_{p \in \mathbb{N}^+} Fix(T^p) = \text{ periodic points.}$$

For $y \in Per(T)$:

$$\mathbb{P}_y := \{\, F \in Lip(X, \mathbb{R}) \mid \exists F - \text{maxim. meas. supported on } \mathcal{O}(y) \,\}$$

$\overset{\circ}{\mathbb{P}}_y := \text{int } \mathbb{P}_y$ on $Lip(X, \mathbb{R})$ i.e. $\mathcal{O}(y)$ is stably the maximizing measure.

Proposition

Let $F, u \in Lip(X,\mathbb{R})$ *with* $\mathcal{L}_F(u) = u$,
$\overline{F} := F + \alpha(F) + u - u \circ T$, *and* $M \in \mathbb{N}^+$.

Suppose that
$\exists \delta_k \downarrow 0 \quad \exists\, p_k$*-periodic* δ_k*-pseudo-orbit* $(x_i)_{i=1}^{p_k}$
in $[\overline{F} = 0]$,
with at most M *jumps*,

such that for $\gamma_k := \min_{1 \leqslant i < j \leqslant p_k} d(x_i, x_j)$,

$$\lim_k \frac{\gamma_k}{\delta_k} = +\infty.$$

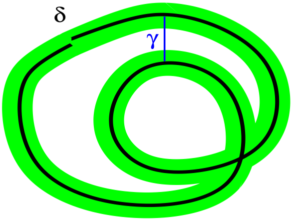
Then

$$F \in closure\Big(\bigcup_{y \in Per(T)} \overset{\circ}{\mathbb{P}}_y\Big).$$

Idea of the Proof:

1. Close the pseudo-orbit using the shadowing lemma.
2. Subtract a channel: $G(x) = F(x) - \varepsilon d(x, \mathcal{O}(y))$.
3. Will prove that any calibrating pre-orbit for G has α-limit $= \mathcal{O}(y)$. →23
4. Each time a calibrating pre-orbit separates from $\mathcal{O}(y)$ the action of $\overline{G}$ diminishes by a fixed amount.
5. Total action of a calibrating orbit is finite $\Longrightarrow$ expends finite time far from $\mathcal{O}(y)$.
6. (expansivity) $\Longrightarrow$ α-limit $= \mathcal{O}(y)$.

$\lim_n \frac{\gamma_n}{\delta_n} = +\infty.$



We close a pseudo-orbit in $[\overline{F} = 0]$.

Size of the jumps $\delta_n \approx$ the action of the shadowing closed orbit $\mathcal{O}(y)$.

Distance of the approaches $(\delta_n \ll)\gamma_n \approx$ how much action is lost

$$G(x) = F(x) - \varepsilon\, d(x, \mathcal{O}(y))$$

when a G-calibrating pre-orbit separates from $\mathcal{O}(y)$.

Proof of the Perturbation Proposition

Let $x_1, \dots, x_p$ be a δ-pseudo-orbit in $[\overline{F} = 0]$ with at most M jumps and minimal approach $\min_{i,j} d(x_i, x_j) \geqslant \gamma$.

$\mathcal{O}(y) = \{y_i\}_{i=1}^p$ closed orbit which shadows $\{x_i\}_{i=1}^p$.

Shadowing Lemma $\implies A_{\overline{F}}(\mathcal{O}(y)) = \sum_{i=1}^p \overline{F}(y_i) \geqslant -K\,\delta$.

Perturbation $G(x) = F(x) - \ \varepsilon\, g(x) + \beta$, $\quad g(x) := d(x, \mathcal{O}(y))$,

$$\beta :=\alpha(\overline{F} - \varepsilon g) = - \sup_{\mu \in \mathcal{M}(T)} \int (F - \varepsilon\, g)\; d\mu$$

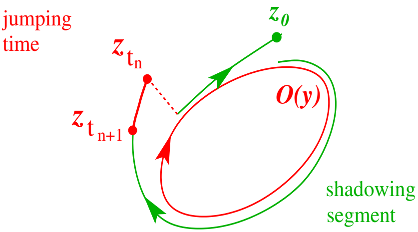
$$\beta \leqslant -A_{\overline{F}}(\mu_y) \leqslant -\frac{K\delta}{p}. \qquad \text{(on supp } \mu_y\text{: } g \equiv 0)$$

v = Calibrated sub-action for G: $\mathcal{L}_G(v) = v$.

Let $\{z_k\}_{k \leqslant 0}$ be a calibrating pre-orbit for G.

$0 > t_1 > t_2 > \cdots$ Jump times when the pre-orbit z_k separates from $\mathcal{O}(y)$:

$$d(z_{t_n}, \mathcal{O}(y)) \geqslant \rho, \qquad \rho \approx \delta \ll \gamma.$$



On a shadowing segment

$$\left| \sum_{t_{n+1}-1}^{t_n-1} G(z_k) - \underbrace{\sum_{corresp} G(y_k)}_{\text{each period sums} \leqslant 0} \right| \leqslant Lip(G) \sum_{i=0}^{\infty} \lambda^i \rho \leqslant K\, \rho.$$

$$\sum_{t_{n+1}-1}^{t_n-1} G(z_k) \leqslant \underbrace{K\, \delta}_{\substack{\text{remainder} \\ \leqslant \text{ one period}}} + K\, \rho.$$

At the jump (when $t_{n+1} < t_n - 1$)

$$G(z_{t_{n+1}}) \leqslant \overline{F}(z_{t_{n+1}}) - \varepsilon\, d(z_{t_{n+1}}, \mathcal{O}(y)) + \beta$$
$$\leqslant 0 - \varepsilon\,\gamma + \frac{K\delta}{p}$$

When $d(z_m, \mathcal{O}(y)) < \rho$ but not the first jump also estimate $G(z_m) < 0$.

Adding:

$$\sum_{t_{n+1}}^{t_n - 1} G(z_k) \leqslant \boxed{-\varepsilon\gamma} + 2K\delta + K\rho, \qquad \rho \approx \delta \ll \gamma.$$
$$< b < 0.$$

On a calibrating pre-orbit

$$v(z_{-N}) = v(z_0) + \sum_{k=-N}^{-1} G(z_k).$$

But v is (Lipschitz) continuous on $X \implies$ bounded.

Each shadowing segment adds $< b < 0$.

$\implies$ finitely many jumps.

$\implies$ The α-limit of the calibrating orbit $\{z_n\}$ is the periodic orbit $\mathcal{O}(y)$. $\square$

Back to zero entropy 29

Proof of the Main Theorem

We prove that $\mathcal{O} := \bigcup_{y \in Per(T)} \overset{\circ}{\mathbb{P}}_y$ is open and dense in $Lip(X, \mathbb{R})$. It is clearly open.

- Argument by Contradiction.

Suppose it is not dense. Then there is an open subset $\emptyset \neq \mathcal{U} \subset Lip(X, \mathbb{R})$ disjoint from $\mathcal{O}$.

By Morris Theorem we can choose $F \in \mathcal{U}$ such that there is a unique (ergodic) F-maximizing measure μ and

$$h_\mu(T) = 0.$$

μ maximizing $\Longrightarrow$ for any calibrating sub-action u,
$\text{supp}(\mu) \subset [\overline{F} = 0]$.

(here $\overline{F} = F + \alpha + u - u \circ T$ depends on u)

μ is ergodic $\Longrightarrow$ there is a generic point q for μ,
i.e. for any continuous function $f : X \to \mathbb{R}$

$$\int f \, d\mu = \langle f \rangle(q) = \lim_N \frac{1}{N} \sum_{i=0}^{N-1} f(T^i(q)).$$

Many returns

Since we are arguing by contradiction.
By the perturbation proposition with $M = \#jumps = 2$, there is $Q > 0$ and $\delta_0 > 0$ such that if $0 < \delta < \delta_0$,

- $(x_k)_{k \geqslant 0} \subset \mathcal{O}(q)$ is a p-periodic δ-pseudo-orbit
- with at most 2 jumps,
- made with elements of the positive orbit of q (which is in $[\overline{F} = 0]$).

Then

$$\gamma = \min_{1 \leqslant i < j < p} d(x_i, x_j) < \tfrac{1}{2} Q \delta.$$

i.e. every closed pseudo-orbit in $\mathcal{O}(q)$ with at most 2 jumps must have an intermediate return with proportion at most $\frac{1}{2}Q$.

Main idea: This will contradict the zero entropy of μ.

Fix a point $w \in \mathrm{supp}(\mu)$ for which Brin-Katok theorem holds:

$$h_\mu(T) = -\lim_{L \to +\infty} \frac{1}{L} \log \mu\big(V(w, L, \varepsilon)\big),$$

where $V(w, L, \varepsilon)$, $L \in \mathbb{N}$, $\varepsilon > 0$ is the dynamic ball

$$V(w, L, \varepsilon) := \{\, x \in X \mid d(T^k x, T^k w) < \varepsilon,\ \forall k = 0, \ldots, L \,\}.$$

Since T is an expanding map, for $\varepsilon < e_0$ small we have

$$V(w, L, \varepsilon) = S_1 \circ \cdots S_L\big(B(T^L w, \varepsilon)\big),$$

for an appropriate sequence of inverse branches S_i.
Thus

$$V(w, L, \varepsilon) \subseteq B(w, \lambda^L \varepsilon).$$

Main idea:

The measure of $V(w, L, \varepsilon)$ can be estimated by the proportion of the orbit of q which is spent on it.

approximating the characteristic function by a continuous fn.

If the measure of $V(w, L, \varepsilon)$ decreases exponentially with L it contradicts $h_\mu(T) = 0$.

We estimate the measure of the ball $B(w, \lambda^L \varepsilon) \supset V(w, L, \varepsilon)$.

Using the perturbation proposition we shall see that: Two consecutive visits of the orbit of q in the ball $B(w, \lambda^L \varepsilon)$ give rise to (exponentially) many intermediate returns (or approximations) which are outside the ball.

Thus the measure of the ball decreases exponentially with L.

Let N_0 be such that $2Q^{-N_0} < \delta_0$.
For $N > N_0$ let $0 \leqslant t_1^N < t_2^N < \cdots$ be all the Q^{-N} returns to w, i.e.

$$\{t_1^N, t_2^N, \dots\} = \{\, n \in \mathbb{N} \mid d(T^n q, w) \leqslant Q^{-N} \,\}.$$

q =Generic point. w = Brin-Katok point.

Proposition

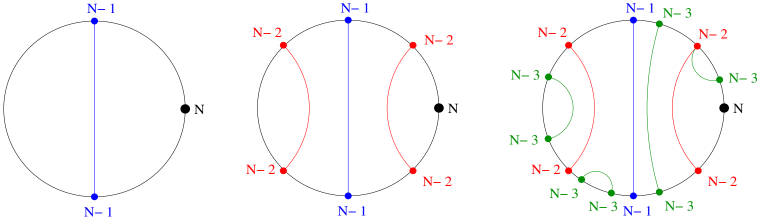
For any $\ell \geqslant 1$, $\quad t_{\ell+1}^N - t_\ell^N \geqslant \sqrt{2}^{N-N_0-1}$.

From this

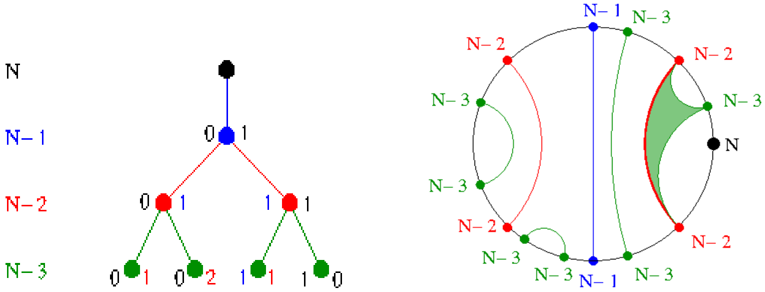
$$\mu\big(B(w, Q^{-N})\big) \leqslant \frac{1}{\sqrt{2}^{N-N_0-1}}.$$

And then $\mu\big(V(w, L, \varepsilon)\big) \leqslant \mu\big(B(w, \lambda^L \varepsilon)\big)$ decreases exponentially with L.
This contradicts the zero entropy.

Inductive process



A cascade of approaches implies by the inductive process



An example of a distribution of returns implied by the perturbation lemma and the tree representing it.

Want to estimate the length of an orbit segment with a return of size Q^{-N} and show that it grows exponentially with N.

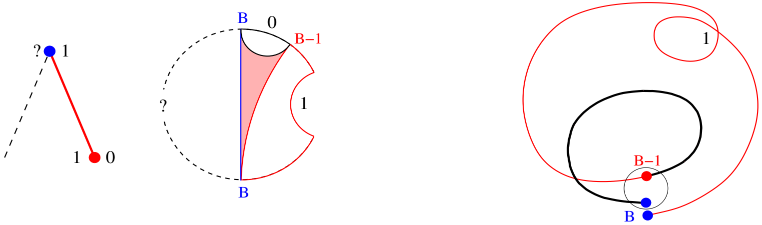
2 ways of counting:

- Count black nodes = when the end point of a new approach was not counted before.
- Count branches of the tree using

Lemma

If $K = [\overline{F} = 0]$ has no periodic points then $\exists \delta_0 > 0 \ \ \forall \delta \in [0, \delta_0[$ s.t. any pseudo-orbit in K with $\leqslant 2$ jumps has length at least 100.

length of the pseudo orbits are $\geqslant$ 100, don't care much if we counted the endpoints.

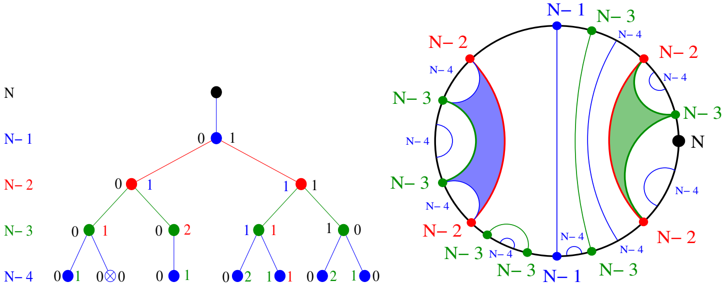


The black approach is a $B-1$ approach. But the red approach is also a $B-1$ approach because the implied approach is of size $\frac{1}{2}Q^{-B+1}$ and

$$\frac{1}{2}Q^{-B+1} + Q^{-B} < Q^{-B+1}.$$

So we draw the red line and shadow the triangle.

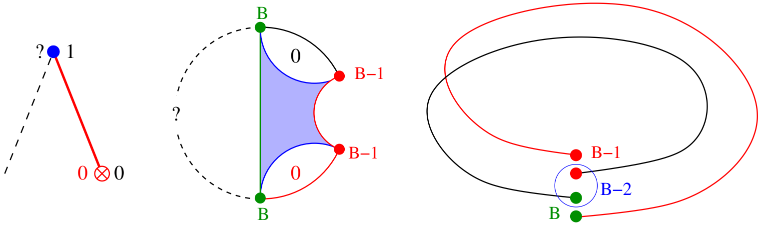
The two "sides" of the triangle are new closed pseudo-orbits.



$$t^N_{\ell+1} - t^N_\ell \geqslant \#\{\text{ black nodes in the tree }\}.$$

$\bullet$ = black node, $\otimes$ = white node.

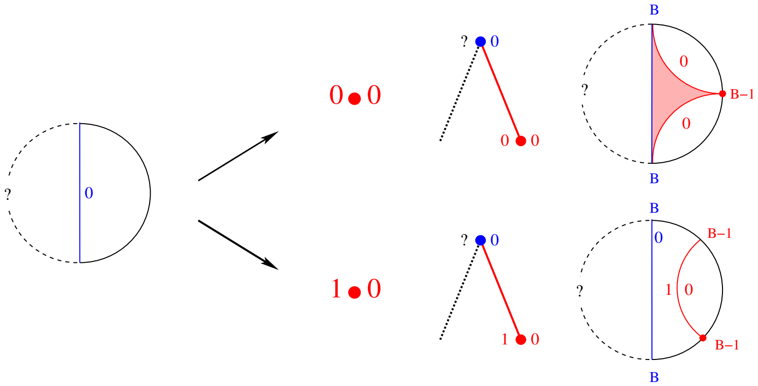
The square



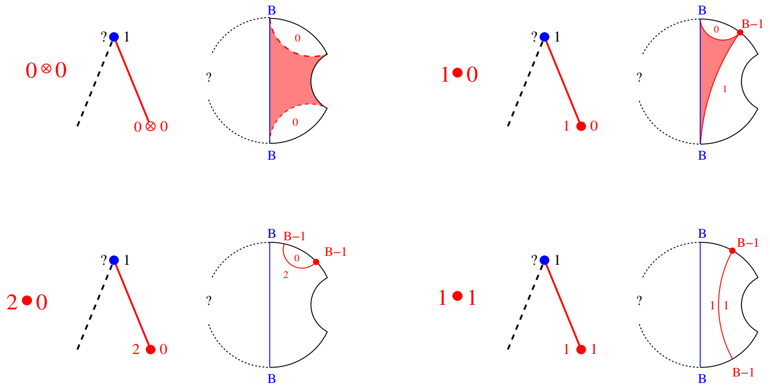
When both endpoints of the new $B-2$ approach are endpoints of previous approaches. Then the four endpoints are $B-2$ approaches because

$$\frac{1}{2}Q^{-B+2} + Q^{-B+1} < Q^{-B+2}$$
$$\frac{1}{2}Q^{-B+2} + Q^{-B} < Q^{-B+2}.$$

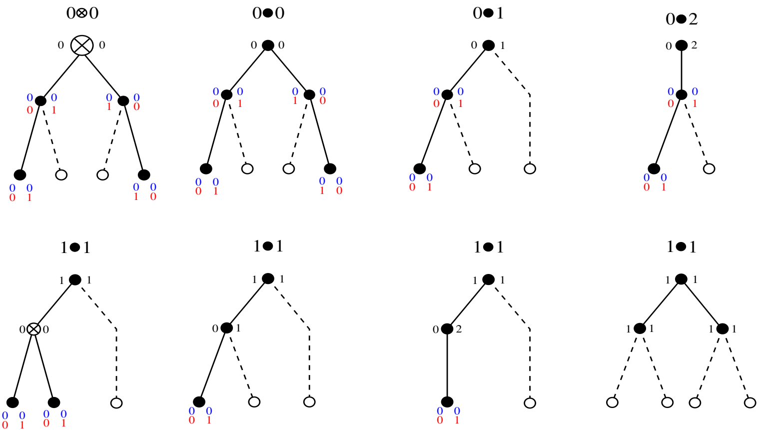
We draw both lines and shadow the rectangle.



Possible nodes ending a branch with a label 0,
i.e. child pseudo-orbits of a periodic 1-pseudo-orbit with only one jump.



All possible nodes ending a branch with a label 1,
i.e. child specifications of a periodic 1-specification with two jumps.
A white dot 0 ⊗ 0 or a 2 ● 0 (3-jump) is always followed by at least one approach with a 0 (1-jump) which re-starts the duplication process.



Possible 2-steps in the tree. They have:

- At least two black nodes in levels $N-1$, $N-2$.
- At least two ending branches at level $N-2$.

$\Longrightarrow$ there is duplication of points every two levels: exponential growth with rate $\sqrt{2}$.

- The process continues as long as $Q^{-M} < \delta_0$, i.e. $N_0 < M < N$.
- The number of nodes duplicates every 2 steps in the tree.

$$\#\{ \text{ black nodes } \} \geqslant 2^{\frac{N-N_0-1}{2}} = \sqrt{2}^{N-N_0-1}.$$