BEYOND BOWEN LECTURE 6: THE ENTROPY GAP IN NONPOSITIVE CURVATURE

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Recap of lecture 5, and introduction to lecture 6.

For the geodesic flow on a rank 1 non-positive curvature manifold, we have stated and discussed our main results on uniqueness of equilibrium states, and the K property for these equilibrium states. Our results hold under the hypothesis of the pressure gap $P(\mathrm{Sing}, \varphi) < P(\varphi)$. Thus, being able to verify the pressure gap is of central importance for our results. We demonstrate how to prove the entropy gap $h(\mathrm{Sing}) < h(X)$ using a direct argument based on the specification property. The full argument can be found in [BCFT18]. The entropy gap was originally proved by Knieper as a corollary of his result that the MME is unique [Kni98].

When $\mathrm{Dim}(M) = 2$, the entropy gap holds because $h_{\mathrm{top}}(\mathrm{Sing}) = 0$, which can be observed through the following simple argument:

Let $\mu \in \mathcal{M}^e(\mathrm{Sing})$. By Ruelle inequality, $h_\mu \leq \lambda^+(\mu) = \int -\varphi^u d\mu = 0$. The last equality is because $\varphi^u \equiv 0$ on Sing for a surface. By the variational principle, $h_{\mathrm{top}}(\mathrm{Sing}) = \sup\{h_\mu : \mu \in \mathcal{M}^e(\mathrm{Sing})\} = 0$.

However, in higher dimensions, it is not at all clear a priori that the entropy gap should always hold. The Gromov example demonstrates that starting in $\mathrm{Dim}(M) = 3$, we may have $h_{\mathrm{top}}(\mathrm{Sing}) > 0$.

The advantage of our approach is that is is constructive, suitable for generalization, and sheds light on the 'entropy gap' phenomenon. Today, we present the basic idea behind using the specification property to produce entropy, and then demonstrate how we can use this approach for geodesic flow in non-positive curvature.

1. Warm-up: shifts with specification

The basic mechanism for using specification to produce entropy is simply to construct exponentially many orbit segments "by hand". This idea can be seen in its simplest form in the following result, which has been known since the 70's, see [DGS76].

Theorem 1.1. *A non-trivial symbolic space (Σ, σ) with the specification property has positive entropy.*

Here, non-trivial just means that Σ is not a single fixed point for σ. For example, the space $\Sigma = \{0^\infty\}$ trivially has the specification property. By specification, we mean the strong version: there exists $\tau \geq 0$ so for all $w^1, w^2 \in \mathcal{L} = \mathcal{L}(\Sigma)$, there exists a word v of length exactly τ so that $w^1 v w^2 \in \mathcal{L}$.

Proof. For some n, we take $w^1, w^2 \in \mathcal{L}_n$ with $w^1 \neq w^2$. For each $k \geq 1$ and each $\underline{i} \in \{1,2\}^k$, define

$$\Phi(\underline{i}) \in \mathcal{L}(\Sigma)$$

by

$$\Phi(\underline{i}) = w^{i_1} v^1 w^{i_2} v^2 \cdots v^{k-1} w^{i_k},$$

where all the v^j have length τ and the expression on the right hand side is chosen to be in the language of Σ. The existence of such a word is guaranteed by the specification property.

Since $w^1 \neq w^2$, we can see that Φ is injective on $\{1,2\}^k$.

Thus $\#\mathcal{L}_{nk+(k-1)\tau}(\Sigma) \geq 2^k$.

Thus, $h(\Sigma) \geq \lim_{k\to\infty} \frac{1}{k(n+\tau)} \log 2^k = \frac{1}{n+\tau} \log 2 > 0$. □

We take this basic idea further, and sketch a proof of the following result about shifts with specification. The statement is not that interesting but the proof contains the main entropy production idea that we will use.

Theorem 1.2. *Consider a symbolic space (Σ, σ) with the specification property. Let $Y \subset \Sigma$ be a compact invariant proper subset. Then $h(Y) < h(X)$.*

We use the specification property, words in $\mathcal{L}(Y)$ and a single word $w \notin \mathcal{L}(Y)$ to construct at least $e^{n(h(Y)+\epsilon)}$ words in $\mathcal{L}_n(X)$ for large n, giving the desired result.

We fix $w \notin \mathcal{L}(Y)$ with length t. We fix a window size $n > t + 2\tau$ and consider N such windows. Consider in each window a word from the language of Y, i.e. a word of the form $y_{[1,n]} \in \mathcal{L}_n(Y)$. We can perform the following 'surgery' to create a word which is in $\mathcal{L}_n(X)$ but not $\mathcal{L}(Y)$:

$$y_{[1,n]} \to y_{[1,n-t-2\tau]} v^1 w v^2,$$

where the words v^1, v^2 of length τ are chosen as needed for the specification property. We can consider N windows, and a word $y_{[1,nN]} \in \mathcal{L}_{nN}(Y)$. In each of the N windows of length n, we can decide whether to do surgery or not. Given this choice, we use the specification property to create a new word. In this way, from a single word $y_{[1,nN]}$, we can create $2^N - 1$ new words of length nN in $\mathcal{L}(X)$ by varying over all the possible choices of windows for doing this surgery procedure.

This looks promising; however, it is too naive: we have to be careful as we vary over $y_{[1,nN]} \in \mathcal{L}(Y)$. In any window we selected for surgery, we are losing all the information on the last $t + 2\tau$ entries in the window. This means that at worst $\#\mathcal{L}_{t+2\tau}$ distinct words could be mapped to the same word for EACH window we select for surgery. If we select too many windows, the gain in new words is far outweighed by the loss coming from this multiplicity estimate.

Fix: We carry out surgery on a small proportion of the windows, and argue that the number of new words created beats the loss of multiplicity. We give some details.

Fix $\alpha > 0$ small. Consider the $N-1$ internal boundary points of the N windows, i.e. the set

$$A = \{n, 2n, 3n, \dots, (N-1)n\}$$

Declare a proportion $\alpha > 0$ of the points in A to be "on", and let $J \subset A$ be the "on" set. Let

$$\underline{J}_N^\alpha = \{J \subset A : \#J = \alpha N - 1\},$$

where we assume for notational simplicity that $\alpha N \in \mathbb{N}$. Note that

$$\#\underline{J}_N^\alpha = \binom{N-1}{\alpha N - 1} \geq \alpha e^{(-\alpha \log \alpha) N}.$$

Fix $y = y_{[1,nN]} \in \mathcal{L}_{nN}(Y)$. We carry out our surgery procedure on the windows whose boundaries are determined by J[1] . We obtain a new word $\Phi_J(y) \in \mathcal{L}_{nN}(X)$ which is definitely not in $\mathcal{L}(Y)$.

The set $\{\Phi_J(y) : J \in \underline{J}_N^\alpha\}$ is disjoint. We carry out this procedure for each word in $\#\mathcal{L}_{nN}(Y)$ and each $J \in \underline{J}_N^\alpha$. This gives

$$\#\left(\bigcup_{y_{[1,nN]} \in \mathcal{L}_{nN}(Y)} \bigcup_J \Phi_J(y)\right) \geq (C^{-1})^{\alpha N - 1} \binom{N-1}{\alpha N - 1} \#\mathcal{L}_{nN}(Y),$$

where $C = \#\mathcal{L}_{t+2\tau}(Y)$ is the upper bound on the number of words in Y that we lose the ability to distinguish at a single surgery site. Note that C is independent of α and N. This gives

$$\#\mathcal{L}_{nN}(X) \geq \alpha e^{(-\alpha \log \alpha) N} e^{-\alpha N \log C} \#\mathcal{L}_{nN}(Y).$$

We obtain

$$h(X) \geq h(Y) + \frac{\alpha}{n}(-\log \alpha - \log C).$$

If $\alpha > 0$ is chosen small enough, the second term is positive.

[1]Each such window determined by the choice of J has length some multiple of n. The procedure is to remove the last $t + 2\tau$ symbols from each window and replace with a word of the form $v^1 w v^2$ where the words v^j are provided by the specification property to ensure that the procedure creates a word in $\mathcal{L}_{nN}(X)$

2. Entropy gap for geodesic flow

We prove that for the geodesic flow on $X = T^1M$ for a closed rank 1 non-positive curvature manifold M, that the entropy gap $h_{\text{top}}(X) > h_{\text{top}}(\text{Sing})$ holds.

We follow the same entropy production strategy that we sketched previously. The singular set $\text{Sing} \subset X$ is a compact invariant proper subset. But how should we construct orbits? We do not expect that orbit segments contained in Sing will have the specification property. For example, orbit segments which are contained in the interior of a flat strip definitely do not have the specification property because of the flat geometry. If we stay ϵ-close inside the flat strip on the time interval $[0, t]$, the amount of additional time needed to escape the flat strip grows with t.

So we want to use a specification argument on orbit segments without specification. Let us recall what kind of orbits DO have specification: it suffices to know that both the start and end of the orbit segment are "uniformly" in the regular set.

More precisely, for any $\eta > 0$, we have the specification property on the collection

$$\mathcal{C}(\eta) = \{(x, t) : x, f_t x \in \text{Reg}(\eta)\},$$

where $\text{Reg}(\eta) = \{x : \lambda(v) \geq \eta\}$. See yesterday's lecture for the definition of λ and discussion of why the specification property holds on $\mathcal{C}(\eta)$. We require a reasonable way to approximate orbit segments in Sing by orbit segments in $\mathcal{C}(\eta)$.

2.1. Approximating singular orbits by regular orbits.

We define a map

$$\Pi_t : \text{Sing} \to \text{Reg}.$$

Very roughly, our slogan (which doesn't make sense as a rigorous statement) is:

"Move the start of (v, t) along its stable into $\text{Reg}(\eta)$. Move the end along an unstable into $\text{Reg}(\eta)$"

We now explain the construction that makes this idea precise. In our approximation of (v, t), we ask that:

(1) $\Pi_t(v), \Pi_t(f_t v) \in \text{Reg}(\eta)$.
(2) there exists L so $\Pi_t(f_s v)$ and Sing are close for $s \in [L, t - L]$.

In the second property, one might hope to find L so $\Pi_t(f_s v)$ and $f_s v$ are close for $s \in [L, t - L]$; however, this is too much to ask for. We can see the issue if (v, t) is in the middle of a flat strip; the best we can hope for is that $\Pi_t(v)$ approaches the *edge* of the flat strip.

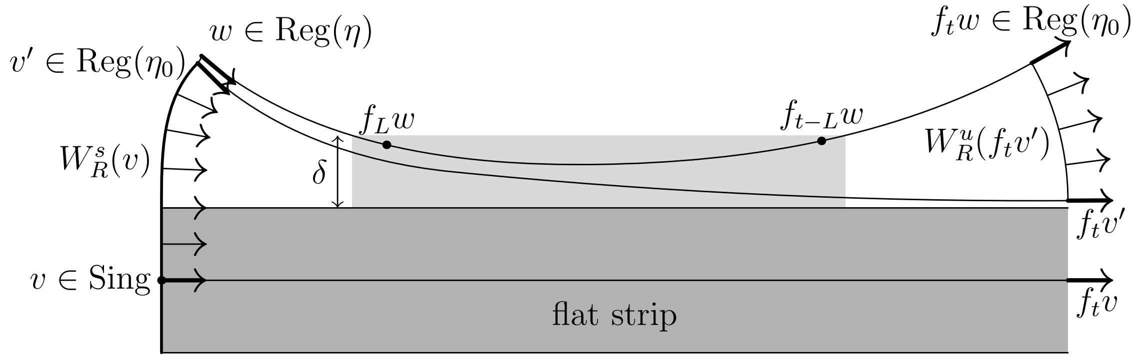
We fix η_0 so $\text{Reg}(\eta_0)$ has nonempty interior.

Claim: There exists $R > 0$ such that for for every $v \in T^1M$ we have both $W^s_R(v) \cap \text{Reg}(\eta_0) \neq \emptyset$ and $W^u_R(v) \cap \text{Reg}(\eta_0) \neq \emptyset$.

The claim follows from the density of stable and unstable leaves, and a compactness argument.

Given $v \in \mathrm{Sing}$, choose $v' \in W^s_R(v) \cap \mathrm{Reg}(\eta_0)$. Then for $f_t(v')$, choose $f_t(w) \in W^u_R(f_t v') \cap \mathrm{Reg}(\eta_0)$. Define $\Pi_t(v) := w$.

By continuity of λ, we have $\lambda(w) \geq \eta$ for an η slightly smaller than η_0. We can argue that the function $\lambda^u(f_t w)$ is small along all of the orbit segment except for an initial and terminal run of uniformly bounded length. This in turn implies that $d(f_t w, \mathrm{Sing})$ is small, giving us condition (2).



In conclusion, we obtain the following properties:

Theorem 2.1. *For every $\delta > 0$ and $\eta \in (0, \eta_0)$, there exists $L > 0$ such that for every $v \in \mathrm{Sing}$ and $t \geq 2L$, the image $w = \Pi_t(v)$ has the following properties:*

(1) *$w, f_t(w) \in \mathrm{Reg}(\eta)$;*
(2) *$d(f_s(w), \mathrm{Sing}) < \delta$ for all $s \in [L, t-L]$;*
(3) *for every $s \in [L, t-L]$, $f_s(w)$ and v lie in the same connected component of $B(\mathrm{Sing}, \delta) := \{w \in T^1M : d(w, \mathrm{Sing}) < \delta)\}$.*

Claim: The collection $\{(\Pi_t(v), t) : v \in \mathrm{Sing}\}$ has specification. This is because an orbit segment $(\Pi_t(v), t)$ both starts and ends in $\mathrm{Reg}(\eta)$. As discussed, the collection of such orbit segments has the specification property.

We certainly do not expect the map Π_t to preserve separation of orbits. For example, in the picture above, we would expect a $v' \in \mathrm{Sing}$ defining a parallel geodesic (for example the arrow just above v in the picture) to be mapped to the same (or similar) point. However, we can argue that Π_t has bounded multiplicity on a (t, ϵ) separated set, independent of t, in the following sense:

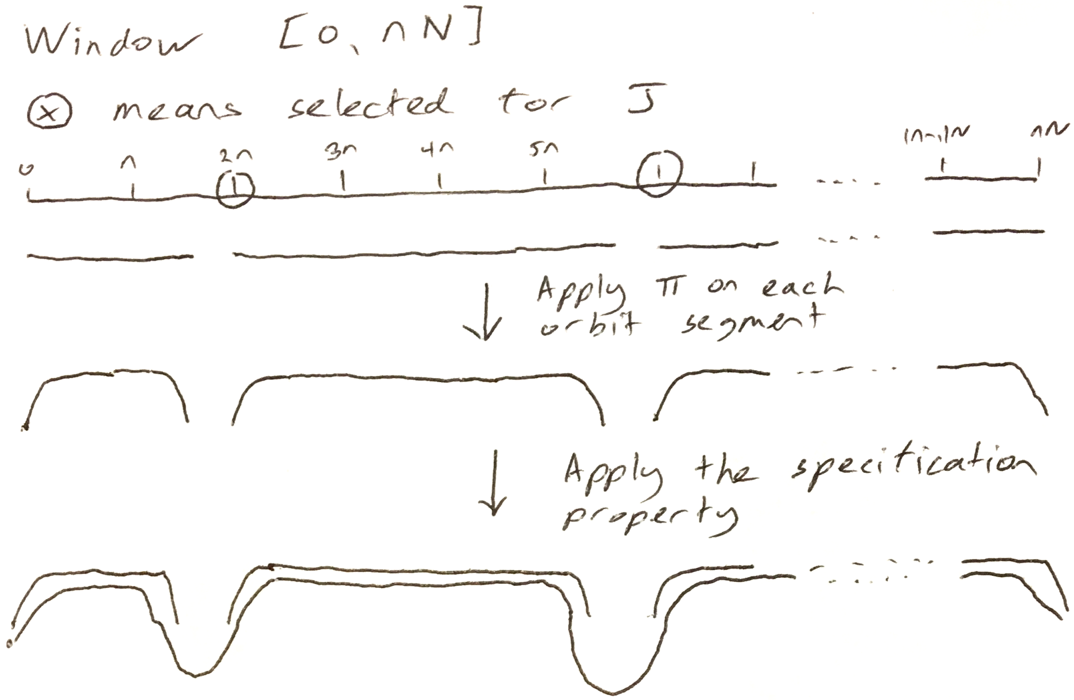
Proposition 2.2. *For every $\epsilon > 0$, there exists $C > 0$ such that if $E_t \subset \mathrm{Sing}$ is a $(t, 2\epsilon)$-separated set for some $t > 0$, then for every $w \in T^1M$, we have $\#\{v \in E_t \mid d_t(w, \Pi_t v) < \epsilon\} \leq C$.*

This is obtained using estimates in the universal cover. We omit the details.

Now let us return to our entropy production argument. It is basically the argument we saw in the previous section. We consider a time window $[0, nN]$.

We choose a subset J of $\alpha N - 1$ elements from the set $\{n, 2n, 3n, \dots, (n-1)N\}$. We let $l_1, l_2, \dots, l_{\alpha N}$ for the lengths of the intervals (in order) whose endpoints are determined by J.

For $(v_1, v_2, \dots, v_{\alpha N}) \in \text{Sing}^{\alpha N}$, we apply the map $\Pi_{l_i - T}$ to each coordinate and glue the orbit segments we obtain using specification (where T is the transition time in the specification property at a suitable scale).



We run this construction over $(g_{l_i - T}, \epsilon)$-separated sets for Sing in each coordinate, and for each choice of J.

We construct exponentially more orbits than there are in Sing. The argument is analogous to our previous entropy production argument: for $\alpha > 0$ small, the growth from the $\binom{N-1}{\alpha N - 1}$ term beats the loss coming from multiplicity in the construction.

We conclude that $h(X) > h(\text{Sing})$ as required.

The approach developed here generalizes to give a pressure gap for the following class of potentials, see [BCFT18]:

Theorem 2.3: Pressure gap (BCFT)

Consider the geodesic flow over a closed rank one manifold of nonpositive curvature. Let $\varphi\colon T^1M \to \mathbb{R}$ be a continuous function that is locally constant on a neighborhood of Sing. Then $P(\mathrm{Sing}, \varphi) < P(\varphi)$.

2.2. **Other applications of pressure production.** The argument for entropy and pressure production is quite flexible, and can be used in many other contexts. For example, in [CT13] we used a variation on this argument to show that for a continuous potential φ with the Bowen property on the β-shift,

$$\limsup \frac{1}{n} \sum_{i=1}^{n-1} \varphi(\sigma^i w^\beta) < P(\Sigma_\beta, \varphi),$$

where w^β is the lexicographically maximal sequence in Σ_β.

Another variation of the argument can be used to prove that a unique equilibrium state μ_φ coming from Bowen's theorem (i.e. from the assumptions of expansivity, specification and the Bowen property) satisfies

$$P(\varphi) > \sup_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu,$$

and thus that the entropy of μ is positive[2]. Such a potential is often called *hyperbolic*. This idea was developed and extended recently in the symbolic setting in [CC].

3. Conclusion

In this series of three lectures, for the geodesic flow over a closed rank one manifold of nonpositive curvature, and potentials which are Hölder or a multiple of the geometric potential, we demonstrated that the pressure gap implies uniqueness of the equilibrium state, and that the equilibrium state has the K property. We gave a direct constructive proof of the entropy gap. This immediately gives the pressure gap for the potentials $q\varphi^u$ for q small. For surfaces, the pressure gap holds for $q\varphi^u$ for $q < 1$. It is an open question whether this is always true in higher dimensions.

The techniques presented here apply in a wide range of other settings in smooth and symbolic dynamics. Vaughn Climenhaga surveyed some of these applications last week. We expect that the techniques will continue to find new applications in other classes of examples beyond uniform hyperbolicity. We hope that these notes will aid the interested reader in adapting these tools to new classes of examples.

[2] `https://vaughnclimenhaga.wordpress.com/2017/01/26/entropy-bounds-for-equilibrium-states/`

References

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