BEYOND BOWEN LECTURE 5: UNIQUENESS OF EQUILIBRIUM STATES (CONTINUED); MIXING PROPERTIES

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Recap of lecture 4, and introduction to lecture 5.

Last time, we introduced our preliminaries on the geodesic flow on a rank 1 non-positive curvature manifold, stated our main results, and described how to obtain a decomposition for the space of orbit segments using a function λ which measures curvature of horospheres. Today, we go into more detail on precisely what hypotheses must be verified to obtain uniqueness of the equilibrium state.

It is well known that a unique equilibrium state must be ergodic. We discuss how to improve ergodicity to much stronger mixing properties: the Kolmogorov K-property and the Bernoulli property.

1. General theorem for uniqueness of equilibrium states

We now rigorously state the abstract theorem from [CT16] that we use to prove our uniqueness results.

Theorem 1.1: Uniqueness of equilibrium states beyond Bowen (Climenhaga-T.)

Let $(X, \mathcal{F})$ be a flow on a compact metric space, and $\varphi : X \to \mathbb{R}$ be a continuous potential function. Suppose that $P^{\perp}_{\text{exp}}(\varphi) < P(\varphi)$ and $X \times [0, \infty)$ admits a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ with the following properties:

(I) $\mathcal{G}$ has specification;
(II) φ has the Bowen property on $\mathcal{G}$;
(III) $P([\mathcal{P}] \cup [\mathcal{S}], \varphi) < P(\varphi)$.

Then $(X, \mathcal{F}, \varphi)$ has a unique equilibrium state μ_φ.

Many of the ideas and definitions needed to understand what this theorem says will be familiar from earlier lectures. So, I will emphasize the main novel points. For completeness, all the definitions in the above statement follow in §1.1.

The first point we should discuss is the expansivity condition $P^{\perp}_{\text{exp}}(\varphi) < P(\varphi)$. The notion of entropy of obstructions to expansivity for discrete-time maps were introduced previously. This definition is for *flows*, and also incorporates a potential φ. It captures the topological pressure of points with 'non-expansive behavior' in the sense of flows.

Given a potential φ, the *pressure of obstructions to expansivity* is $P^\perp_{\exp}(\varphi) := \lim_{\epsilon\to 0} P^\perp_{\exp}(\varphi,\epsilon)$, where

$$P^\perp_{\exp}(\varphi,\epsilon) = \sup_{\mu\in\mathcal{M}^e(\mathcal{F})}\left\{h_\mu(f_1) + \int \varphi\,d\mu \;:\; \mu(\mathrm{NE}(\epsilon,\mathcal{F})) = 1\right\},$$

where $\mathrm{NE}(\epsilon,\mathcal{F})$ is the non-expansive set for the flow (at scale ϵ). For an expansive map, the set of points that stay close to x is by definition only the point x itself.

For an expansive flow, the set of points that stay close to a point x for all time is an orbit segment of x. Our set of non-expansive points for a flow is defined accordingly. For $x \in X$ and $\epsilon > 0$, we let the *bi-infinite Bowen ball* be

$$\Gamma_\epsilon(x) = \{y \in X \;:\; d(f_t x, f_t y) \le \epsilon \text{ for all } t \in \mathbb{R}\}.$$

The *set of non-expansive points at scale* ϵ is

$$\mathrm{NE}(\epsilon,\mathcal{F}) := \{x \in X \mid \Gamma_\epsilon(x) \not\subset f_{[-s,s]}(x) \text{ for any } s > 0\},$$

where $f_{[a,b]}(x) = \{f_t x : a \le t \le b\}$.

The other major new ingredient is the regularity assumption on φ. We say that $\varphi\colon X \to \mathbb{R}$ has the *Bowen property on* $\mathcal{C} \subset X \times [0,\infty)$ if there are $\epsilon, K > 0$ such that for all $(x,t) \in \mathcal{C}$ and $y \in B_t(x,\epsilon)$, we have $|\Phi(x,t) - \Phi(y,t)| \le K$, where $\Phi(x,t) = \int_0^t \varphi(f_s x)\,ds$.

If φ has the Bowen property on $\mathcal{C} = X \times [0,\infty)$, then our definition agrees with the original definition of Bowen.

If $(X,\mathcal{F})$ is uniformly hyperbolic, a standard argument based on exponential expansion/contraction and the local product structure shows that any Hölder continuous φ is Bowen on $X \times [0,\infty)$.

We can prove the Bowen property on $\mathcal{G}$ for Hölder continuous φ using essentially the same argument.

We also show that the geometric potential φ^u is Bowen on $\mathcal{G}$. This is one of the hardest parts of the analysis of [BCFT18], and relies on detailed estimates involving the Riccati equation.

Finally, we point out two things which do NOT matter for us. First, the reader might be curious about the notation $[\mathcal{P}]$, which we chose to be reminiscent of the integer part. We needed to consider $[\mathcal{P}]$ instead of $\mathcal{P}$ for an abstract decomposition for a step in the proof that requires a passage from continuous to discrete time. However, this distinction is irrelevant for the decomposition considered here (or indeed any λ-decomposition). Second, the reader might be wondering about questions of coarse scale, which were emphasized and used very effectively in Vaughn's lectures. Issues of coarse scale do not arise here. We ask for, and obtain, the specification property at arbitrarily small scales. This removes a great deal of technicality from the analysis.

1.1. **All the definitions for the abstract theorem (Optional).** For completeness, we collect all the definitions used in Theorem 1.1 more formally. Given a flow $(X, \mathcal{F})$, we think of $X \times [0, \infty)$ as the space of finite-length orbit segments by identifying (x, t) with $\{f_s(x) : 0 \leq s < t\}$. Given $\mathcal{C} \subset X \times [0, \infty)$ and $t \geq 0$ we let $\mathcal{C}_t = \{x \in X : (x, t) \in \mathcal{C}\}$. The partition function associated to $\mathcal{C}$ is

$$\Lambda(\mathcal{C}, \varphi, \delta, t) = \sup\left\{\sum_{x \in E} e^{\Phi(x,t)} : E \subset \mathcal{C}_t \text{ is } (t, \delta)\text{-separated}\right\}.$$

The *pressure of* φ *on* $\mathcal{C}$ is

$$P(\mathcal{C}, \varphi) = \lim_{\delta \to 0} \limsup_{t \to \infty} \frac{1}{t} \log \Lambda(\mathcal{C}, \varphi, \delta, t).$$

For $\mathcal{C} = \emptyset$ we then define $P(\emptyset, \varphi) = -\infty$.

A collection of orbit segments $\mathcal{C} \subset X \times [0, \infty)$ has *specification at scale* $\rho > 0$ if there exists $\tau = \tau(\rho)$ such that for every $(x_1, t_1), \ldots, (x_N, t_N) \in \mathcal{C}$ there exist a point $y \in X$ and times $\tau_1, \ldots, \tau_{N-1} \in [0, \tau]$ such that for $s_0 = \tau_0 = 0$ and $s_j = \sum_{i=1}^{j} t_i + \sum_{i=1}^{j-1} \tau_i$, we have

$$f_{s_{j-1}+\tau_{j-1}}(y) \in B_{t_j}(x_j, \rho)$$

for every $j \in \{1, \ldots, N\}$. A collection $\mathcal{C} \subset X \times [0, \infty)$ has *specification* if it has specification at all scales. If $\mathcal{C} = X \times [0, \infty)$ has specification, then we say the flow has specification.

We prove a stronger version of this property for the collection $\mathcal{C}$ defined in Lecture 4, in which the conclusion that 'there exist a point y and times $\tau_1, \ldots, \tau_{N-1} \in [0, \tau]$' is replaced with the conclusion that 'for *every* collection of times $\tau_1, \ldots, \tau_{N-1}$ with $\tau_i \geq \tau$ for all i, there exists a point y'. That is, we are able to take all the transition times to be exactly τ, or any length at least τ that we choose.

A *decomposition for* $X \times [0, \infty)$ consists of three collections $\mathcal{P}, \mathcal{G}, \mathcal{S} \subset X \times [0, \infty)$ for which there exist three functions $p, g, s \colon X \times [0, \infty) \to [0, \infty)$ such that for every $(x, t) \in X \times [0, \infty)$, the values $p = p(x, t)$, $g = g(x, t)$, and $s = s(x, t)$ satisfy $t = p + g + s$, and

$$(x, p) \in \mathcal{P}, \quad (f_p(x), g) \in \mathcal{G}, \quad (f_{p+g}(x), s) \in \mathcal{S}.$$

The conditions we are interested in depend only on the collections $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ rather than the functions p, g, s. However, we work with a fixed choice of (p, g, s) for the proof of the abstract theorem to apply.

For a collection $\mathcal{P}$, we define

$$[\mathcal{P}] := \{(x, n) \in X \times \mathbb{N} : (f_{-s}x, n + s + t) \in \mathcal{P} \text{ for some } s, t \in [0, 1]\}$$

and similarly for $[\mathcal{S}]$. The reason that in general we control the pressure of $[\mathcal{P}] \cup [\mathcal{S}]$ rather than the collection $\mathcal{P} \cup \mathcal{S}$ is a consequence of a technical step in the proof of the abstract result in [CT16] that required a passage from continuous to discrete time. This distinction does not matter for us because for a λ-decomposition we can show by a simple argument that for all $\epsilon > 0$, $P([\mathcal{B}(\eta)], \varphi) \leq P(\mathcal{B}(\eta + \epsilon), \varphi)$.

2. Kolmogorov property for equilibrium states

Now we discuss the Kolmogorov property for equilibrium states. The following result is recent work, and not on the arXiv yet (I am happy to share an early version of the manuscript on request).

Theorem 2.1: K property (Call-T.)

Let (g_t) be the geodesic flow over a closed rank 1 manifold M and let $\varphi\colon T^1M \to \mathbb{R}$ be $\varphi = q\varphi^u$ or be Hölder continuous. If $P(\mathrm{Sing}, \varphi) < P(\varphi)$, then the unique equilibrium state μ_φ has the Kolmogorov property.

Note that when $\mathrm{Dim}(M) = 2$, it is already known that μ_φ is Bernoulli by applying the work of Ledrappier-Lima-Sarig [LLS16] which uses countable state symbolic dynamics for 3-dimensional flows.

Mixing for μ_{KBM} was already known in all dimensions by Babillot [Bab02]. In higher dimensions, the K-property is a new result even for the MME μ_{KBM}.

We recall the hierarchy of mixing properties (this is an "express train" version of the hierarchy):

Bernoulli $\Rightarrow$ K $\Rightarrow$ mixing of all orders $\Rightarrow$ mixing $\Rightarrow$ weak mixing $\Rightarrow$ ergodic

The K property has a few equivalent formulations. It has a nice characterization in terms of entropy as follows: We say μ has the K *property* if any non-trivial partition ξ (i.e. $\xi \neq \{\emptyset, X\}$ mod 0 measure sets) has positive entropy $h(\mu, \xi) > 0$. The *Pinsker σ-algebra* for an invariant measure μ is essentially the biggest σ-algebra with entropy 0. Thus, μ has the K property if and only if the Pinsker σ-algebra for μ is trivial.

The implications above are not 'if and only if"s in general. However, in smooth settings with some hyperbolicity, a classic strategy for proving the Bernoulli property is to move *up* the hierarchy, establishing K, and then proving that K implies Bernoulli. This approach was notably carried out by Ornstein and Weiss [OW73, OW98], Pesin [Pes77], and Chernov and Haskell [CH96]. In particular, a major success of Pesin theory is his proof that that Liouville measure restricted to the regular set is Bernoulli.

It is natural for us to follow the same approach[1]. I won't say much about this because this is the part of the project we are still writing up, but we think it is safe to announce our result at least in the MME case; I will be cautious about claiming the result for equilibrium states, but I can certainly say that this is the goal of our project and that we believe we are on track to announce it soon.

Theorem 2.2: Bernoulli property (Call-T.)

Let (g_t) be the geodesic flow over a closed rank 1 manifold M. The unique measure of maximal entropy μ_{KBM} is Bernoulli.

3. Tools to prove Theorem 2.1

Our main tool is a fantastic result of Ledrappier [Led77], which has been 'rediscovered' in the last couple of years. The proof is about one page long, and gives an insightful criteria for the K property in terms of thermodynamic formalism. The original result is for discrete-time systems. We state here a version of it for flows[2]. The idea is to consider the product flow $(X \times X, \mathcal{F} \times \mathcal{F})$, i.e. the flow $(f_s \times f_s)_{s \in \mathbb{R}}$ given by

$$(f_s \times f_s)(x, y) = (f_s x, f_s y) \text{ for } s \in \mathbb{R}.$$

Theorem 3.1: Criteria for K property (After Ledrappier)

Let $(X, \mathcal{F})$ be a flow such that f_t is asymptotically entropy expansive for all $t \neq 0$, and let φ be a continuous function on X. Let $(X \times X, \mathcal{F} \times \mathcal{F})$ be the product of two copies of $(X, \mathcal{F})$.

We define the function $\Phi : X \times X \to \mathbb{R}$ by $\Phi(x_1, x_2) = \varphi(x_1) + \varphi(x_2)$.

If Φ has a unique equilibrium measure in $\mathcal{M}(X \times X, \mathcal{F} \times \mathcal{F})$, then the unique equilibrium measure for φ in $\mathcal{M}(X, \mathcal{F})$ has the Kolmogorov property.

The system $(X, \mathcal{F}, \varphi)$ must have a unique equilibrium state because of the following simple lemma.

Lemma 3.2. *Let μ be an equilibrium state for $(X, \mathcal{F}, \varphi)$. Then $\mu \times \mu$ is an equilibrium state for $(X \times X, \mathcal{F} \times \mathcal{F}, \Phi)$.*

Proof. Observe that

$$h_{\mu \times \mu}(f_1 \times f_1) = h_\mu(f_1) + h_\mu(f_1)$$

[1]Thanks to Omri Sarig, Todd Fisher, and Ali Tahzibi for encouraging us to not stop at K!!

[2]The proof of the flow version requires some adaptions from the discrete-time case; it can also be deduced from the discrete-time result via a half-page proof.

and

$$\int \Phi \, d(\mu \times \mu) = \int \varphi \, d\mu + \int \varphi \, d\mu.$$

Therefore, $h_{\mu \times \mu}(f_1 \times f_1) + \int \Phi \, d(\mu \times \mu) = 2P(X, \mathcal{F}, \varphi) = P(X \times X, \mathcal{F} \times \mathcal{F}, \Phi)$. $\square$

Thus if μ, ν are distinct equilibrium states for $(X, \mathcal{F}, \varphi)$, then $\mu \times \mu$ and $\nu \times \nu$ are both equilibrium states for Φ. If Φ has a unique equilibrium state, then this means that $\mu \times \mu = \nu \times \nu$ and hence $\mu = \nu$; thus, we get uniqueness of the equilibrium state downstairs, and we see that if Φ has a unique equilibrium state, it must have the form $\mu \times \mu$ where μ is the unique equilibrium state for φ.

We sketch the main idea of Ledrappier's result. By the argument above, if Φ has a unique equilibrium state, then so does φ. Write μ for this measure. As we observed, $\mu \times \mu$ is an equilibrium state for Φ (the unique one, by assumption). Now assume that μ is not K. Then μ has a non-trivial Pinsker σ-algebra. This can be used to define another equilibrium state for Φ. Contradiction.

Strategy of proof of Theorem 2.1. Given Ledrappier's result, and the general thrust of these lectures, the strategy for proving the K property is now clear. We want to show that the product system of two copies of the geodesic flow has a unique equilibrium state for the class of potentials under consideration.

So let's find a decomposition for the product system.

Problem: Lifting decompositions to products in general does not work well. One fact we do have in our favor is that if $\mathcal{G}$ has good properties, then so does $\mathcal{G} \times \mathcal{G}$. However, we need $\mathcal{G} \times \mathcal{G}$ to arise in a decomposition for $(X \times X, \mathcal{F} \times \mathcal{F})$. For the decompositions we defined for our symbolic examples, this does not look at all promising. Exercise: Try to do it for S-gap shifts. You'll see the issue.

Idea: Work with a nice class of decompositions that DOES behave well under products. We claim that the class of λ-decompositions is such a class. We used λ-decompositions already in most of our applications: Mañé examples, partially hyperbolic diffeos with 1D center, and geodesic flow on rank one non-positive curvature manifolds. The decomposition used to study geodesic flow on surfaces with no focal points by [CKP18] is also a λ-decomposition.

We state the definition of an 'abstract' λ-decomposition precisely. Let X be a compact metric space, $\mathcal{F} : X \to X$ a continuous flow, and $\varphi : X \to \mathbb{R}$ a continuous potential. Let $\lambda : X \to [0, \infty)$ be a bounded lower semicontinuous function and $\eta > 0$. Let $B(\eta) = \{(x, t) \mid \frac{1}{t} \int_0^t \lambda(f_s(x)) \, ds \leq \eta\}$ and

$$\mathcal{G}(\eta) = \{(x, t) \mid \frac{1}{\rho} \int_0^\rho \lambda(f_s(x)) \, ds \geq \eta \text{ and } \frac{1}{\rho} \int_0^\rho \lambda(f_{-s} f_t(x)) \, ds \geq \eta \text{ for } \rho \in [0, t]\}.$$

Let $\mathcal{P} = \mathcal{S} = B(\eta)$, and let $\mathcal{G} = \mathcal{G}(\eta)$. We define a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ as follows. Given an orbit segment $(x, t) \in X \times [0, \infty)$, we decompose (x, t) by taking

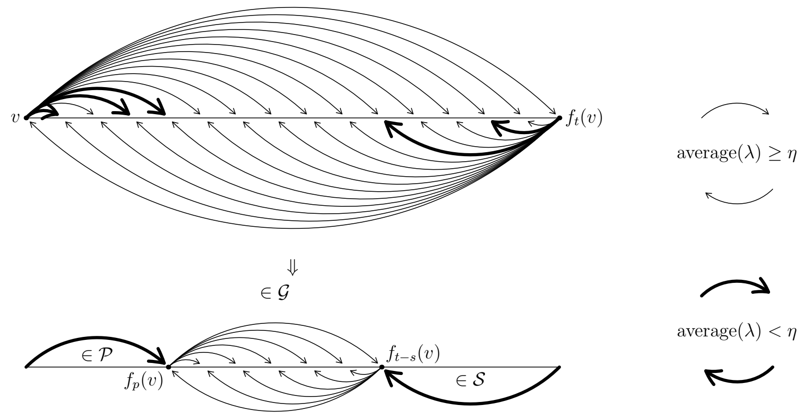

FIGURE 1. Decomposing an orbit segment (again).

the longest initial segment in $\mathcal{P}$ as the prefix, and the longest terminal segment which lies in $\mathcal{S}$ as the suffix. The good core is what is left over. We say that a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ defined in this way is a λ-*decomposition* (with constant η). We ask that the function λ is lower semi-continuous since this allows both continuous functions as well as indicator functions of open sets.

In order to show that $(X \times X, \mathcal{F} \times \mathcal{F})$ has a unique equilibrium state, we need to find a decomposition for the product system. When $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ is a λ-decomposition, we are able to do this by defining $\tilde{\lambda} : X \times X \to [0, \infty)$ as

$$\tilde{\lambda}(x, y) = \lambda(x)\lambda(y).$$

This function inherits lower semicontinuity from λ, and we consider $\tilde{\lambda}$-decompositions for $(X \times X, \mathcal{F} \times \mathcal{F})$. That is, for $\eta > 0$, we let

$$\tilde{B}(\eta) = \{(x, y, t) \mid \frac{1}{t} \int_0^t \tilde{\lambda}(f_s x, f_s y)\, ds \leq \eta\},$$

and we let

$$\tilde{\mathcal{G}}(\eta) = \{(x, y, t) \mid \frac{1}{\rho} \int_0^\rho \tilde{\lambda}(f_s x, f_s y)\, ds \geq \eta, \frac{1}{\rho} \int_0^\rho \tilde{\lambda}(f_{t-s} x, f_{t-s} y)\, ds \geq \eta \text{ for } \rho \in [0, t]\}.$$

The collections $\tilde{\mathcal{P}} = \tilde{\mathcal{S}} = \tilde{B}(\eta)$, and $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}(\eta)$ define a $\tilde{\lambda}$-decomposition $(\tilde{\mathcal{P}}, \tilde{\mathcal{G}}, \tilde{\mathcal{S}})$.

The $\tilde{\lambda}$-decomposition ensures that λ is uniformly positive in both coordinates for an orbit segment in $\tilde{\mathcal{G}}$. This means that the arguments for specification and the Bowen property carry over to $\tilde{\mathcal{G}}$.

But how big are $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{S}}$? If $\lambda = 0$ on one of the coordinates, then anything is allowed on the other. Roughly, we can show that:

$$P(\tilde{\mathcal{P}} \cup \tilde{\mathcal{S}}, \varphi) \approx P(\varphi) + P(\mathcal{P} \cup \mathcal{S}).$$

Recall that $P(\Phi) = 2P(\varphi)$. Thus, if we have $P(\mathcal{P} \cup \mathcal{S}) < P(\varphi)$, then we expect to be able to obtain the estimate $P(\tilde{\mathcal{P}} \cup \tilde{\mathcal{S}}, \varphi) < P(\Phi)$. This is the strategy we carry out in our setting.

3.1. **Expansivity issues.** Unfortunately, specification and regularity are not the whole story in the flow case. In fact, it is dealing with continuous time, and related expansivity issues that are the difficult point in our analysis.

Recall that for flows we define

$$\mathrm{NE}(\epsilon, \mathcal{F}) := \{x \in X \mid \Gamma_\epsilon(x) \not\subset \text{ orbit segment } \}.$$

For a products of flows, the set $\Gamma_\epsilon(x, y)$ always contains $f_{[-s,s]}x \times f_{[-s,s]}x$. That is, we are considering a flow with a 2-dimensional center. The theory presented at the start of this lecture does not apply directly. We have to build a new theory that controls

$$\mathrm{NE}^\times(\epsilon) := \{(x, y) \in X \times X \mid \Gamma_\epsilon(x, y) \not\subset f_{[-s,s]}(x) \times f_{[-s,s]}(y) \text{ for any } s > 0\}.$$

There are no new difficulties with counting estimates, but serious issues arise when we build adapted partitions. In the discrete time case, our adapted partition elements look like pixels and can be used to approximate sets. In the flow case, our adapted partition elements approach a small piece of orbit, so look like thin cigars. Collections of partition elements can thus be used to approximate flow-invariant sets. In the 'product of flows' case, the best we can do is approximate sets invariant under $f_s \times f_t$ for *all* $s, t \in \mathbb{R}$. This creates new technical obstacles that must be overcome in our uniqueness proof. In particular, to run our ergodicity proof, we need to be able to approximate sets which are invariant only under $f_s \times f_s$ for all $s \in \mathbb{R}$. This is a fundamental additional difficulty.

We get round this by proving weak mixing for μ using a lower joint Gibbs estimate which gives a kind of partial mixing for sets which are flowed out by a small time interval. This can be used to prove weak mixing of μ by a spectral argument. This is equivalent to the desired ergodicity of $\mu \times \mu$.

3.2. **Finally.** We anticipate that our results on K and Bernoulli for geodesic flow in non-positive curvature will be available on the arXiv later this Summer. Ben Call also has results which abstract our technique for obtaining the K-property in both the discrete-time and continuous-time cases. This will provide convenient verifiable conditions for the K property for systems admitting λ-decompositions beyond the geodesic flow in nonpositive curvature. These results, and their application to many of the other systems admitting λ-decompositions discussed in these lectures, are in progress and will be announced a little further down the road.

References

[Bab02] Martine Babillot, *On the mixing property for hyperbolic systems*, Israel J. Math. **129** (2002), 61–76. MR 1910932

[BCFT18] K. Burns, V. Climenhaga, T. Fisher, and D. J. Thompson, *Unique equilibrium states for geodesic flows in nonpositive curvature*, Geom. Funct. Anal. **28** (2018), no. 5, 1209–1259.

[CH96] N. I. Chernov and C. Haskell, *Nonuniformly hyperbolic K-systems are Bernoulli*, Ergodic Theory Dynam. Systems **16** (1996), no. 1, 19–44. MR 1375125

[CKP18] Dong Chen, Lien-Yung Kao, and Kiho Park, *Unique equilibrium states for geodesic flows over surfaces without focal points*, arXiv preprint arXiv:1808.00663 (2018).

[CT16] Vaughn Climenhaga and Daniel J. Thompson, *Unique equilibrium states for flows and homeomorphisms with non-uniform structure*, Adv. Math. **303** (2016), 745–799.

[Led77] François Ledrappier, *Mesures d'équilibre d'entropie complètement positive*, Dynamical systems, Vol. II—Warsaw, Soc. Math. France, Paris, 1977, pp. 251–272. Astérisque, No. 50. MR 0486419

[LLS16] François Ledrappier, Yuri Lima, and Omri Sarig, *Ergodic properties of equilibrium measures for smooth three dimensional flows*, Comment. Math. Helv. **91** (2016), no. 1, 65–106.

[OW73] Donald S. Ornstein and Benjamin Weiss, *Geodesic flows are Bernoullian*, Israel J. Math. **14** (1973), 184–198. MR 0325926

[OW98] Donald Ornstein and Benjamin Weiss, *On the Bernoulli nature of systems with some hyperbolic structure*, Ergodic Theory Dynam. Systems **18** (1998), no. 2, 441–456. MR 1619567

[Pes77] Ja. B. Pesin, *Characteristic Ljapunov exponents, and smooth ergodic theory*, Uspehi Mat. Nauk **32** (1977), no. 4 (196), 55–112, 287. MR 0466791