BEYOND BOWEN'S SPECIFICATION PROPERTY LECTURE 4: GEODESIC FLOWS IN NONPOSITIVE CURVATURE

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Recap of week 1 talks, and introduction to week 2.

We recall some of the highlights of Vaughn Climenhaga's talks from Week 1. Vaughn's talks focused on the theory for *measures of maximal entropy (MME)*. In particular, he presented Bowen's approach to the uniqueness of the measure of maximal entropy, and the recent extensions of this approach. He showed how new developments in the theory apply to a wide class of systems including β-shifts, partially hyperbolic DA systems, certain billiards, and the class of geodesic flows with no conjugate points on Riemannian surfaces. A key idea is the notion of a decomposition of the space of orbit segments, and some of the applications rely on being able to carry out the specification approach at a coarse scale.

For Week 2, the theory we will discuss is in some ways more general, and in some ways more focused. The additional generality is that we will discuss the theory of uniqueness of *equilibrium states* rather than just uniqueness of MME. The new focus is that all week we will consider the setting of geodesic flow on a closed manifold of nonpositive curvature. This is often considered to be the primary class of non-uniformly hyperbolic flows in the dynamics literature. In order to study this class, we will need to introduce tools and techniques that apply in far greater generally. However, our goal of studying geodesic flow in non-positive curvature is our main motivation, and this informs our choice of exposition.

DISCLAIMER: While we include some references, these notes make no attempt to give a complete account of the current literature in the field, which is vibrant and continually growing. These are preliminary notes, and we sometimes adopt a conversational writing style. I hope that the informal style will be helpful for current purposes; I invite the reader to look at our original papers [CT16, BCFT18] for a more careful and precise account. I also disclaim that these notes have not yet been proofread to the level of a research paper, and some sections were typed up while "Flying the Friendly Skies" as United Airlines calls it. All six lectures will be polished up later into one coherent document with typos fixed and exposition streamlined. If you notice any typos in this write-up, or see some prose that is confusing, please do not hesitate to let me know.

1. Geodesic flow in nonpositive curvature

We collect some geometric preliminaries [1]. For more details, we recommend recent works [BCFT18, GS14], and more classical references [Bal95, Ebe01, Ebe96] Let $M = (M^n, g)$ be a closed connected C^∞ Riemannian manifold with nonpositive sectional curvature and dimension n, and $(g_t)_{t\in\mathbb{R}}$ denote the geodesic flow on the unit tangent bundle $X = T^1M$. The geodesic flow is defined by picking a point and a direction (i.e. an element of T^1M), and walking at unit speed along the geodesic determined by that data. Geodesic flows are of central importance in the theory of dynamical systems, and encode many important features of the geometry and topology of the underlying manifold M. There is a natural volume measure on X called the Liouville measure. We denote it μ_L. Locally, μ_L is the product of the Riemannian volume on M and Haar measure on the unit sphere of dimension $n-1$.

For purposes of exposition, we will often think about the surface case $n = 2$, although our approach applies in higher dimension too. We can think about a genus 2 torus with an embedded flat cylinder, and negative curvature elsewhere, as a first example. We could also think about the example of a genus 2 torus where the curvature vanishes around a single closed geodesic, and is strictly negative elsewhere. Of course, much more complex examples exist. Another important example to keep in mind is the Gromov example in dimension 3.

The geodesic flow in nonpositive curvature is a primary example of non-uniform hyperbolicity. Morally, the difficult phenomenon that one has to deal with here is the co-existence of high complexity (roughly corresponding to negative curvature) and low complexity dynamical behavior (roughly corresponding to zero curvature) in the same system. We split up the phase space according to this dichotomy. That is, we decompose the unit tangent bundle as

$$T^1M = \mathrm{Reg} \sqcup \mathrm{Sing},$$

where $v \in \mathrm{Sing}$ if there exists a parallel orthogonal Jacobi field (defined below), and $v \in \mathrm{Reg}$ otherwise.

In the surface case, let K be the Gauss curvature, and for $v \in T^1M$, let $\pi(v)$ be its footpoint in M. We have the following nice and understandable criteria for $v \in \mathrm{Sing}$:

In $\mathrm{Dim}(M) = 2$, $v \in \mathrm{Sing}$ if and only if $K(\pi(g_tv)) = 0$ for all $t \in \mathbb{R}$.

That is, Sing is the set of v for which the corresponding geodesic γ_v experiences 0 curvature for all time. For v to belong to Reg, all that is required is that $K(\pi(g_tv)) < 0$ for SOME t. We can see that $v \in \mathrm{Reg}$ may experience arbitrarily weak expansion/contraction because the geodesic can be arranged to experience 0 curvature for a long time (e.g. wrapping round an embedded flat cylinder) before hitting any negative curvature.

[1] Some of the definitions are taken verbatim from [BCFT18] for notational consistency.

The set Sing is closed and flow-invariant, while the set Reg is open and dense in T^1M. We say M is *rank 1* if Reg $\neq \emptyset$, and we assume throughout that M is rank 1. Rank 1 is the typical situation (as demonstrated by the higher rank rigidity theorem of Ballmann, Burns-Spatzier). The rank 1 condition is easy to understand in the surface case. It is equivalent to asking that the genus of M is at least 2; essentially all that is ruled out is the flat torus.

1.1. **Invariant foliations.** TX has invariant subbundles E^s and E^u which integrate to stable and unstable foliations W^s and W^u. We must be a little careful in defining them: we cannot ask that $W^s(v)$ is the set of points so that $d(f_tv, f_tw) \to 0$ as $t \to \infty$ like we can in the uniformly hyperbolic setting. We must instead allow points that stay bounded distance apart in all forward time. However, this does not work as the definition of W^s because it does not distinguish the stable from the flow direction. To do things properly, there are two approaches:

Local approach: Use stable and unstable orthogonal Jacobi fields to define E^s and E^u locally (see §1.2).

Global approach: Use the boundary at infinity of $\tilde{M}$ and Busemann functions to define stable and unstable horospheres H^s and H^u.

$H^s(v)$ can be constructed as follows. Take a circle of radius r around f_rv. Take the limit of these circles as $r \to \infty$. This defines $H^s(v)$. Then $W^s(v)$ is the vector field of inward facing normal unit vectors along $H^s(v)$. The unstable horosphere is constructed analogously. We can define $E^s(v) = TW^s(v)$ and $E^u(v) = TW^u(v)$.

More precisely, given $v \in T^1M$, stable and unstable horospheres H^s_v and H^u_v can be constructed in the universal cover as follows. For H^s_v, we consider the set of points in $\tilde{M}$ at distance r from g_rv, that is

$$S^r(v,+) = \{x \in \tilde{M} : d_{\tilde{M}}(x, g_rv) = r\},$$

and we take the limit of $S^r(v,+)$ as $r \to \infty$. This defines a hypersurface which contains the point πv. The stable horosphere H^s_v is the projection to M (from $\tilde{M}$) of this hypersurface. The stable manifold W^s_v is the normal unit vector field to H^s_v on the same side as v. For H^u_v, we consider the set of points in $\tilde{M}$ at distance r from $g_{-r}v$, that is

$$S^r(v,-) = \{x \in \tilde{M} : d_{\tilde{M}}(x, g_{-r}v) = r\},$$

and we take the limit of $S^r(v,-)$ as $r \to \infty$. The projection to M of this hypersurface is the stable horosphere H^u_v. The unstable manifold W^u_v is the normal unit vector field to H^u_v on the same side as v. The horospheres are C^2 manifolds, and we can define the stable and unstable subspaces $E^s_v, E^u_v \subset T_vT^1_M$ to be the tangent spaces of W^s_v, W^u_v respectively. The bundles E^s, E^u, which are both globally defined in this way, are respectively called the stable and unstable bundles. The bundles E^s, E^u are invariant, and depend continuously on v, see [Ebe01, GW99].

On Reg, the subbundles yield the expected splitting $E^s \oplus E^u \oplus E^0$.

Finally, we define a function which is of great importance in thermodynamic formalism. The *geometric potential*, is the function which measures infinitesimal volume growth in the unstable distribution:

$$\varphi^u(v) = -\lim_{t\to 0} \frac{1}{t} \log\det(df_t|_{E^u_v}) = -\frac{d}{dt}\Big|_{t=0} \log\det(df_t|_{E^u_v}).$$

The potential φ^u is continuous and globally defined. When M has dimension 2, the function φ^u is Hölder along unstable leaves [GW99]. It is not known whether φ^u is Hölder along stable leaves. In higher dimensions, it is not known whether φ^u is Hölder continuous on either stable or unstable leaves; an advantage of our approach is that we sidestep the question of Hölder regularity for φ^u.

1.2. **Jacobi fields and local construction of stables/unstables (optional).** A *Jacobi field* along a geodesic γ is a vector field along γ satisfying

$$J''(t) + R(J(t), \dot\gamma(t))\dot\gamma(t) = 0, \tag{1.1}$$

where R is the Riemannian curvature tensor on M and $'$ represents covariant differentiation along γ. Equivalently, a Jacobi field is obtained by taking a one-parameter family of geodesics and differentiating in the parameter coordinate.

We often want to remove variations through geodesics in the flow direction from consideration. If $J(t)$ is a Jacobi field along a geodesic γ and both $J(t_0)$ and $J'(t_0)$ are orthogonal to $\dot\gamma(t_0)$ for some t_0, then $J(t)$ and $J'(t)$ are orthogonal to $\dot\gamma(t)$ for all t. Such a Jacobi field is an *orthogonal Jacobi field*.

A Jacobi field $J(t)$ along a geodesic γ is *parallel at* t_0 if $J'(t_0) = 0$. A Jacobi field $J(t)$ is parallel if it is parallel for all $t \in \mathbb{R}$.

We write $\mathcal{J}(\gamma)$ for the space of orthogonal Jacobi fields for γ; given $v \in T^1M$ there is a natural isomorphism $\xi \mapsto J_\xi$ between T_vT^1M and $\mathcal{J}(\gamma_v)$, which has the property that

$$\|df_t(\xi)\|^2 = \|J_\xi(t)\|^2 + \|J'_\xi(t)\|^2. \tag{1.2}$$

An orthogonal Jacobi field J along a geodesic γ is *stable* if $\|J(t)\|$ is bounded for $t \geq 0$, and *unstable* if it is bounded for $t \leq 0$. The stable and the unstable Jacobi fields each form linear subspaces of $\mathcal{J}(\gamma)$, which we denote by $\mathcal{J}^s(\gamma)$ and $\mathcal{J}^u(\gamma)$, respectively. The corresponding stable and unstable subbundles of TT^1M are

$$E^u(v) = \{\xi \in T_v(T^1M) : J_\xi \in \mathcal{J}^u(\gamma_v)\},$$
$$E^s(v) = \{\xi \in T_v(T^1M) : J_\xi \in \mathcal{J}^s(\gamma_v)\}.$$

The bundle E^c is spanned by the vector field V that generates the flow $\mathcal{F}$. We also write $E^{cu} = E^c \oplus E^u$ and $E^{cs} = E^c \oplus E^s$. The subbundles have the following properties (see [Ebe01] for details):

- $\dim(E^u) = \dim(E^s) = n - 1$, and $\dim(E^c) = 1$;
- the subbundles are invariant under the geodesic flow;

- the subbundles depend continuously on v, see [Ebe01, GW99];
- E^u and E^s are both orthogonal to E^c;
- E^u and E^s intersect non-trivially if and only if $v \in \mathrm{Sing}$;
- E^σ is integrable to a foliation W^σ for each $\sigma \in \{u, s, cs, cu\}$.

It is proved in [Bal82, Theorem 3.7] that the foliation W^s is minimal in the sense that $W^s(v)$ is dense in T^1M for every $v \in T^1M$. Analogously, the foliation W^u is also minimal.

2. Pressure and equilibrium states for flows

2.1. **Topological Pressure.** Let X be a compact metric space, $\mathcal{F} = (f_t)$ a continuous flow on X, and $\varphi\colon X \to \mathbb{R}$ a continuous function. We denote the space of $\mathcal{F}$-invariant probability measures on X by $\mathcal{M}(\mathcal{F})$, and note that $\mathcal{M}(\mathcal{F}) = \bigcap_{t\in\mathbb{R}} \mathcal{M}(f_t)$. We denote the space of ergodic $\mathcal{F}$-invariant probability measures on X by $\mathcal{M}^e(\mathcal{F})$.

We recall the definition of the topological pressure of φ with respect to $\mathcal{F}$. For $\epsilon > 0$ and $t > 0$ the *Bowen ball of radius ϵ and order t* is

$$B_t(x, \epsilon) = \{y \in M \mid d(f_s x, f_s y) < \epsilon \text{ for all } 0 \leq s \leq t\}.$$

Given $\epsilon > 0$ and $t \in [0, \infty)$, a set $E \subset X$ is (t, ϵ)-*separated* if for all distinct $x, y \in E$ we have $y \notin \overline{B_t(x, \epsilon)}$.

We write $\Phi(x, t) = \int_0^t \varphi(f_s x)\, ds$ for the integral of φ along an orbit segment of length t. Let

$$\Lambda(\varphi, \epsilon, t) = \sup \left\{ \sum_{x\in E} e^{\Phi(x,t)} \mid E \subset X \text{ is } (t, \epsilon)\text{-separated} \right\}. \tag{2.1}$$

Then the *topological pressure of φ (with respect to $\mathcal{F}$)* is

$$P(\varphi) = \lim_{\epsilon\to 0} \limsup_{t\to\infty} \frac{1}{t} \log \Lambda(\varphi, \epsilon, t).$$

The *variational principle for pressure* states that if X is a compact metric space and $\mathcal{F}$ is a continuous flow on X, then

$$P(\varphi) = \sup_{\mu\in\mathcal{M}(\mathcal{F})} \left\{ h_\mu + \int \varphi\, d\mu \right\}.$$

A measure achieving the supremum is an *equilibrium state for φ*. When $\varphi = 0$, we recover the topological entropy $h(\mathcal{F})$. An equilibrium state for $\varphi = 0$ is called a *measure of maximal entropy.*

If the entropy map $\mu \mapsto h_\mu$ is upper semi-continuous then equilibrium states exist for each continuous potential function. This is the case in our setting since the flow is entropy-expansivity. This is proved using the flat strip theorem in [Kni98].

2.2. **Pressure on the singular set.** Since the singular set Sing is closed and flow-invariant, we can consider the dynamics restricted to Sing. We write $P(\mathrm{Sing}, \varphi)$ for the pressure of φ with respect to $(g_t)|_{\mathrm{Sing}}$. Equivalently, we can define

$$P(\mathrm{Sing}, \varphi) = \sup \left\{ h_\mu + \int \varphi \, d\mu : \mu \text{ is flow-invariant with } \mu(\mathrm{Sing}) = 1 \right\}.$$

2.3. **Pressure of the geometric potential.** We now consider the pressure of the geometric potential. It follows from the Ruelle-Margulis inequality and the Pesin formula, and that $-\int \varphi^u d\mu$ is the sum of the positive Lyapunov exponents for μ, that

$$P(\varphi^u) = 0,$$

and the Liouville measure μ_L is an equilibrium state for φ^u. In the case of negative curvature manifolds, φ^u is Hölder and μ_L is the unique equilibrium state by Bowen's classic work.

2.4. **Pressure and periodic orbits for geodesic flows (optional).** For $a < b$, let $\mathrm{Per}_R(a, b]$ denote the set of closed regular geodesics with length in the interval $(a, b]$. For each such geodesic γ, let $\Phi(\gamma)$ be the value given by integrating φ around γ; that is, $\Phi(\gamma) := \Phi(v, |\gamma|) = \int_0^{|\gamma|} \varphi(f_t v)\, dt$, where $v \in T^1M$ is tangent to γ and $|\gamma|$ is the length of γ. Given $T, \delta > 0$, let

$$\Lambda^*_{\mathrm{Reg}}(\varphi, T, \delta) = \sum_{\gamma \in \mathrm{Per}_R(T-\delta, T]} e^{\Phi(\gamma)}.$$

For a closed geodesic γ, let μ_γ be the normalized Lebesgue measure around the orbit. Here, we are following a notation convention of Katok: when we say a geodesic, we mean oriented geodesic, and we are considering γ as a periodic orbit living in T^1M. We consider the measures

$$\mu^{\mathrm{Reg}}_{T,\delta} = \frac{1}{\Lambda^*_{\mathrm{Reg}}(\varphi, T, \delta)} \sum_{\gamma \in \mathrm{Per}_R(T-\delta, T]} e^{\Phi(\gamma)} \mu_\gamma.$$

We say that *regular closed geodesics weighted by φ equidistribute* to a measure μ if $\lim_{T\to\infty} \mu^{\mathrm{Reg}}_{T,\delta} = \mu$ in the weak* topology for every $\delta > 0$.

3. Main results for week 2

Our main result on uniqueness of equilibrium states for geodesic flow in nonpositive curvature is the following.

Theorem 3.1: Uniqueness of equilibrium states for rank 1 geodesic flow (Burns-Climenhaga-Fisher -T.)

Let (g_t) be the geodesic flow over a closed rank 1 manifold M and let $\varphi\colon T^1M \to \mathbb{R}$ be $\varphi = q\varphi^u$ or be Hölder continuous satisfying the *pressure gap*

$$P(\text{Sing}, \varphi) < P(\varphi).$$

Then φ has a unique equilibrium state μ. This equilibrium state is hyperbolic, fully supported, and is the weak* limit of weighted regular closed geodesics;

In Lecture 5, we will discuss the following new result, which is not yet on the ArXiv (a preliminary version is available on request).

Theorem 3.2: K and Bernoulli properties (Call-T.)

Any unique equilibrium state provided by the above theorem has the K-property. The unique MME μ_{KBM} has the Bernoulli property.

Remarks on Theorem 3.1:

1) If $P(\text{Sing}, \varphi) = P(\varphi)$, i.e. if the pressure gap fails, there is definitely not a unique fully supported equilibrium state.

2) The case $\varphi = 0$ is due to Knieper using a Patterson-Sullivan type construction at the boundary at infinity.

3) As a corollary of 2), it follows that $h_{\text{top}}(\text{Sing}) < h_{\text{top}}(X)$, i.e. that the entropy gap holds. This is non-trivial since in higher dimensions we may have $h_{\text{top}}(\text{Sing}) > 0$. This is demonstrated by the Gromov example. We have a new direct proof of the entropy gap which we will discuss in Lecture 6.

Theorem 3.3: Entropy gap (Knieper; new direct proof from BCFT)

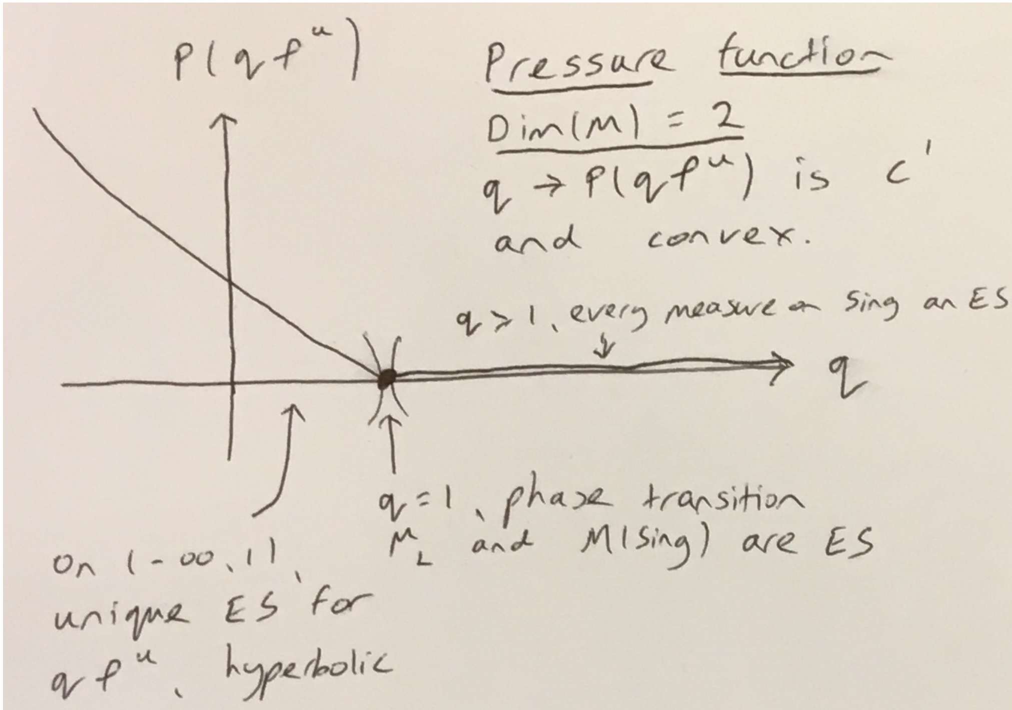
For geodesic flow on a closed rank 1 manifold M, the entropy gap holds

(Actually our proof gives the pressure gap for any continuous potential φ which is locally constant on a a neighbourhood of Sing.)

It follows from the entropy gap and a soft argument based on the variational principle that the pressure gap holds whenever

$$\sup \varphi - \inf \varphi < h_{\text{top}}(X) - h_{\text{top}}(\text{Sing}).$$

This holds, for example, for $q\varphi^u$ with q small.



4) When $\mathrm{Dim}(M) = 2$, φ^u vanishes on Sing and $h_{\mathrm{top}}(\mathrm{Sing}) = 0$.

Thus, $P(\mathrm{Sing}, q\varphi^u) = 0$ for all $q \in \mathbb{R}$.

It is an easy consequence of Ruelle inequality and Pesin formula that

$$P(q\varphi^u) > 0 \text{ for } q < 1.$$

Thus, $q\varphi^u$ has a unique equilibrium state for $q < 1$. We obtain the classic picture of the pressure function in non-uniform hyperbolicity.

4. Proof idea for Theorem 3.1

A key idea is to find a *decomposition* $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ for the space of orbit segments. I'll wait until the next lecture to state our abstract theorem on uniqueness of equilibrium states from [CT16] rigorously, although the idea will be familiar from the MME case which was presented last week. For now, we just sketch the main ideas.

We require that:

(1) $\mathcal{G}$ has good properties: that is, the specification property at all scales, and good control on the integral of φ along orbit segments in $\mathcal{G}$;
(2) The collections $\mathcal{P}, \mathcal{S}$ satisfy $P(\mathcal{P} \cup \mathcal{S}) < P(\varphi)$.

Choose a function $\lambda : X \to [0, \infty)$ which measures 'hyperbolicity'. We want:

(1) λ vanishes on Sing
(2) λ uniformly positive implies uniform estimates.

There is a convenient geometrically-defined function which has the desired properties. We let $\lambda(v)$ be the minimum of the curvature of the stable horosphere $H^s(v)$ and the unstable horosphere $H^u(v)$.

If $v \in$ Sing, then $\lambda(v) = 0$ due to the presence of a parallel orthogonal Jacobi field. The set $\{v \in \text{Reg} : \lambda(v) = 0\}$ may be non-empty, but it has zero measure for any invariant measure [BCFT18, Corollary 3.6].

If $\lambda(v) \geq \eta > 0$, then we have various uniform estimates at the point v, for example on the angle between E^u_v and E^s_v, and on the growth of Jacobi fields at v. Thus, the function λ serves as a useful 'measure of hyperbolicity'. In particular, we get the following distance estimates.

Given $\eta > 0$ and $\delta = \delta(\eta)$ sufficiently small, $v \in T^1M$, and $w, w' \in W^s_\delta(v)$, we have for every $t \geq 0$:

$$d^s(f_t w, f_t w') \leq d^s(w, w') e^{-\int_0^t (\lambda(f_\tau v) - \eta/2)\, d\tau},$$

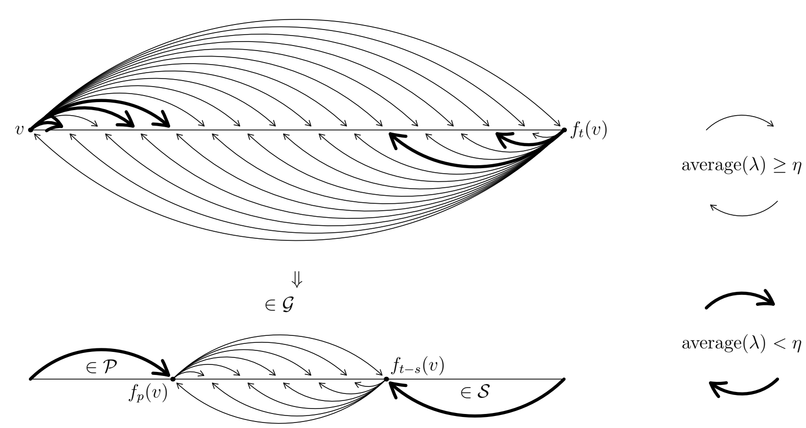
where d^s is the distance on W^s. We get similar estimates for $w, w' \in W^u_\delta(v)$.

We fix $\eta > 0$, and define the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ as follows. Let

$$B(\eta) = \{(x,t) \mid \frac{1}{t}\int_0^t \lambda(f_s(x))\, ds < \eta\},$$

$$\mathcal{G}(\eta) = \{(x,t) \mid \frac{1}{\rho}\int_0^\rho \lambda(f_s(x))\, ds \geq \eta \text{ and } \frac{1}{\rho}\int_0^\rho \lambda(f_{-s} f_t(x))\, ds \geq \eta \text{ for } \rho \in [0,t]\}.$$

Let $\mathcal{P} = \mathcal{S} = B(\eta)$, and let $\mathcal{G} = \mathcal{G}(\eta)$. We decompose an orbit segment (x,t) by taking the longest initial segment in $\mathcal{P}$ as the prefix, and the longest terminal segment which lies in $\mathcal{S}$ as the suffix. The good core is what is left over.



We say that a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ defined in this way by a continuous function $\lambda : X \to [0, \infty)$ is a λ-*decomposition* (with constant η).

This decomposition is useful for us is for the following two reasons:

(1) For $\eta > 0$ small, $P(\mathcal{P} \cup \mathcal{S}, \varphi)$ is close to $P(\mathrm{Sing}, \varphi)$. Thus the pressure gap assumption gives us $P(\mathcal{P} \cup \mathcal{S}, \varphi) < P(X, \varphi)$
(2) $\mathcal{G}$ has the specification property and desired uniform estimates on φ.

We prove that the specification property holds on the collection

$$\mathcal{C}(\eta) = \{(x, t) : \lambda(x) > \eta, \lambda(f_t x) > \eta\}.$$

Clearly, we have $\mathcal{G}(\eta) \subset \mathcal{C}(\eta)$. The proof of the specification property is essentially the one given last week in the uniformly hyperbolic case. The key ingredient is uniformity of the local product structure at the end points of the orbit segments. This is provided by the condition that λ is uniformly positive at these points. Then we use uniform density of unstable leaves to transition between orbit segments. We additionally need some definite contraction along the stable of each orbit segment, which is not hard to obtain.

We also mention another estimate that we hope is illuminating. For an orbit $(x, t) \in \mathcal{G}(\eta)$, the distance estimate

$$d^s(f_t w, f_t w') \le d^s(w, w') e^{-\int_0^t (\lambda(f_\tau x) - \eta/2)\, d\tau}.$$

becomes

$$d^s(f_t w, f_t w') \le d^s(w, w') e^{-t\eta/2}.$$

This is a crucial estimate for controlling the regularity of the potential along the orbit; we will discuss our regularity condition on φ precisely next time.

In summary, the ideas above give us the main ingredients to apply our abstract machinery on uniqueness of equilibrium states. Thus, the pressure gap yields a unique equilibrium state.

4.1. **The function λ in higher dimensions (Optional).** We define a version of the function $\lambda\colon T^1M \to [0, \infty)$ which is suitable for manifolds M with $\mathrm{Dim}(M) \ge 2$. Let H^s, H^u be the stable and unstable horospheres for v. Let $\mathcal{U}^s_v\colon T_{\pi v} H^s \to T_{\pi v} H^s$ be the symmetric linear operator defined by $\mathcal{U}(v) = \nabla_v N$, where N is the field of unit vectors normal to H on the same side as v. This determines the second fundamental form of the stable horosphere H^s. We define $\mathcal{U}^u_v\colon T_{\pi v} H^s \to T_{\pi v} H^s$ analogously. Then $\mathcal{U}^u_v$ and $\mathcal{U}^s_v$ depend continuously on v, $\mathcal{U}^u$ is positive semidefinite, $\mathcal{U}^s$ is negative semidefinite, and $\mathcal{U}^u_{-v} = -\mathcal{U}^s_v$.

For $v \in T^1M$, let $\lambda^u(v)$ be the minimum eigenvalue of $\mathcal{U}^u_v$ and let $\lambda^s(v) = \lambda^u(-v)$. Let $\lambda(v) = \min(\lambda^u(v), \lambda^s(v))$.

The functions λ^u, λ^s, and λ are continuous since the map $v \mapsto \mathcal{U}_v^{u,s}$ is continuous, and we have $\lambda^{u,s} \geq 0$. When M is a surface, the quantities $\lambda^{u,s}(v)$ are just the curvatures at πv of the stable and unstable horocycles, and we recover the definition of λ we stated previously.

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