Klt varieties with trivial canonical class – Holonomy, differential forms, and fundamental groups II
Analytic aspects

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The singular Ricci-flat metric

X projective variety of dimension n, H ample line bundle.

Main assumption

X has canonical singularities and $K_X \sim \mathcal{O}_X$.

Theorem (Eyssidieux-Guedj-Zeriahi '06)

There exists a unique smooth Kähler form ω on X_{reg} such that

1. $\omega \in c_1(H)|_{X_{\mathrm{reg}}}$,
2. $\mathrm{Ric}\,\omega = 0$,
3. $\int_{X_{\mathrm{reg}}} \omega^n = c_1(H)^n$.

Decomposition of the tangent sheaf

Theorem (Greb-Kebekus-Peternell '12, Guenancia '15)

There exists a finite quasi-étale cover $f : A \times Z \to X$ and a decomposition

$$\mathscr{T}_Z = \bigoplus_{i \in I} \mathscr{E}_i$$

such that

1. $\widetilde{q}(Z) = 0$, i.e. $\forall Z' \to Z$ quasi-étale finite, $h^0(Z', \Omega_{Z'}^{[1]}) = 0$.
2. Each $\mathscr{E}_i$ has vanishing c_1, is strongly stable wrt any polarization.
3. The vector bundle $\mathscr{E}_i|_{Z_{\mathrm{reg}}}$ is invariant under parallel transport by ω_Z defined by $f^*\omega = \omega_A \oplus \omega_Z$.

Objectives

Assume $\widetilde{q}(X) = 0$ and decompose $\mathscr{T}_X = \bigoplus_{i \in I} \mathscr{E}_i$ into strongly stable pieces.

Road map

1. Classify/compute $\mathrm{Hol}_\omega(X_{\mathrm{reg}}, \mathscr{E}_i)$.
2. Relate the geometry of $\mathscr{E}_i$ (e.g. global sections of tensor bundles) to its holonomy group $\mathrm{Hol}_\omega(X_{\mathrm{reg}}, \mathscr{E}_i)$.
3. Classify varieties X with $\mathscr{T}_X$ strongly stable (i.e. $|I| = 1$).

Quick recap on riemannian holonomy (1)

(M, g) Riemannian manifold, Levi-Civita connection ∇^g on T_M, $x \in M$.

Parallel transport

Given a loop $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = x$ and $v \in T_{M,x}$, $\exists!$ smooth section $v(t) \in T_{M,\gamma(t)}$ such that

1. $v(0) = v$.
2. $\nabla^g_{\gamma'(t)} v(t) = 0$.

Define $\tau_\gamma(v) := v(1) \in T_{M,x} \rightsquigarrow \tau_\gamma \in \mathrm{O}(T_{M,x}, g_x)$.

Quick recap on riemannian holonomy (2)

Holonomy group

We define

$$G := \mathrm{Hol}(M,g) := \{\tau_\gamma, \gamma \text{ loop at } x\} \subset \mathrm{O}(T_{M,x})$$

and the connected component of the identity

$$G^\circ := \mathrm{Hol}^\circ(M,g) := \{\tau_\gamma, \gamma \text{ loop at } x \text{ homotopic to } 0\} \subset \mathrm{O}(T_{M,x})$$

Link with fundamental group

$G^\circ \triangleleft G$ is normal and $\exists$ canonical surjection $\pi_1(M) \twoheadrightarrow G/G^\circ$.

Quick recap on riemannian holonomy (3)

Main example

(M, ω) Kähler Ricci-flat $\rightsquigarrow G \subset \mathrm{SU}(n)$.

Subbundles

If $E \subset T_M$ is a subbundle invariant by parallel transport, then the holonomy G decomposes as $G_1 \times G_2$ where $G_1 \circlearrowleft E_x$ and $G_2 \circlearrowleft E_x^\perp$ and one sets $\mathrm{Hol}(M, E, g) = G_1$.
If M is complex and E a holomorphic subbundle, then the $\mathscr{C}^\infty$ splitting $T_M = E \oplus E^\perp$ is actually holomorphic.

Elementary results

Assume $\widetilde{q}(X) = 0$ and decompose $\mathscr{T}_X = \bigoplus_{i \in I} \mathscr{E}_i$ into strongly stable pieces of rank r_i, set $G = \mathrm{Hol}(X_{\mathrm{reg}}, \omega)$ for some fixed $x \in X_{\mathrm{reg}}$

Splitting

G decomposes as

$$G = \prod_{i \in I} G_i$$

with $G_i \subset \mathrm{SU}(r_i)$.

Irreducibility vs Stability

The action $G_i \circlearrowleft \mathbb{C}^{r_i}$ (resp $G_i^\circ \circlearrowleft \mathbb{C}^{r_i}$) is *irreducible* iff $\mathscr{E}_i$ is stable (resp. strongly stable) wrt H.

Classification of the restricted holonomy

$\rightsquigarrow$ The standard action of $G_i^\circ = \mathrm{Hol}(X_{\mathrm{reg}}, \mathscr{E}_i, \omega)^\circ$ is irreducible.

Berger-Simons classification

One of the following cases holds

1. $G_i^\circ = \{1\}$.
2. $G_i^\circ = \mathrm{Sp}(r_i/2)$.
3. $G_i^\circ = \mathrm{SU}(r_i)$.

Case 1. $\mathscr{E}_i|_{X_{\mathrm{reg}}}$ is a flat vector bundle: impossible since by Druel's result, it would mean that an abelian variety splits off X (after a finite quasi-étale cover maybe).

Case 2. In the last two cases, G_i/G_i° injects in $\mathrm{U}(1)$; in particular, it is abelian $\rightsquigarrow$ it is finite!

Proof of finiteness claim

By Greb-Kebekus-Peternell, can assume that any linear representation of $\pi_1(X_{\mathrm{reg}})$ extends to $\pi_1(X)$.

$$\begin{tikzcd}
\pi_1(X_{\mathrm{reg}}) \arrow[rr, two heads] \arrow[dr] & & G_i/G_i^\circ \\
& \pi_1(X) \arrow[ur] \arrow[r] & H_1(X,\mathbb{Z}) \arrow[u]
\end{tikzcd}$$

but the first homology group is finite as

$$\begin{aligned}
\mathrm{rank}(H_1(X,\mathbb{Z})) &= \dim_{\mathbb{C}} H^1(X,\mathbb{C}) \\
&= 2\dim_{\mathbb{C}} H^0(X,\Omega_X^{[1]}) = 0
\end{aligned}$$

since $q(X) \le \widetilde{q}(X) = 0$.

Consequence of the classification result

Consequence of the previous results:

The holonomy group of $(X_{\mathrm{reg}}, \omega)$ is known

Up to passing to a further quasi-étale finite cover, one can assume that G_i is either $\mathrm{SU}(r_i)$ or $\mathrm{Sp}(r_i/2)$ and G is the product of these groups.

In particular, one can compute explicitely the algebra of G_i-invariant (resp. G-invariant) vectors under the standard or tensor representations of $\mathscr{E}_{i,x}$ (resp. $\mathscr{T}_{X,x}$).

The result

Theorem: the Bochner principle (Greb-G-Kebekus '17)

Given a linear representation ρ of $\mathrm{GL}(n, \mathbb{C})$, the evaluation map at x induces a bijection between $H^0(X_{\mathrm{reg}}, \mathscr{T}_X^{\rho})$ and $\mathscr{T}_{X,x}^{\rho(G)}$.

Idea of proof

In the smooth case, if σ is a holomorphic tensor, then Bochner-Weitzenböck formula reads

$$\Delta_\omega |\sigma|_\omega^2 = |\nabla^\omega \sigma|_\omega^2$$

since $\mathrm{Ric}\,\omega = 0$. As the integral of a Laplacian is zero, we get $\nabla^\omega \sigma = 0$, i.e. σ is parallel, i.e. σ_x is G-invariant.

The result

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Calabi-Yau and Irreducible Holomorphic Symplectic varieties

If $G = \mathrm{SU}(n)$ or $G = \mathrm{Sp}(n/2)$, then

$$\bigoplus_{p=0}^{n} H^0(X, \Omega_X^{[p]}) = \bigoplus_{p=0}^{n} (\Lambda^p\ (\mathbb{C}^n)^*)^G = \begin{cases} \mathbb{C}[\Omega],\ \Omega = \text{triv. of } K_X. \\ \mathbb{C}[\sigma],\ \sigma = \text{ symplectic 2-form.} \end{cases}$$

and the same holds for any finite, quasi-étale cover $Y \to X$.

Two corollaries

Strongly stable varieties are CY or IHS

Assume $n \geq 2$, and $\mathscr{T}_X$ is strongly stable wrt H. Then, there is a finite, quasi-étale cover $Y \to X$ such either

1. Y is CY variety (i.e. $G = \mathrm{SU}(n)$) or
2. Y is an IHS variety (i.e. $G = \mathrm{Sp}(n/2)$).

The next corollary plays a key role in Höring-Peternell's proof of the decomposition theorem.

Symmetric power of the tangent sheaf.

Assume $n \geq 2$, and $\mathscr{T}_X$ is strongly stable wrt H. Then, for any $r \geq 1$, the sheaf $\mathrm{Sym}^{[r]}\mathscr{T}_X$ is strongly stable wrt H.