The decomposition theorem: the smooth case

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The decomposition theorem

This introductory talk is devoted to the history of the following theorem:

Decomposition theorem

Let M be a compact Kähler manifold with $c_1(M) = 0$ in $H^2(M, \mathbb{R})$.

There exists $M' \to M$ finite étale with $M' = T \times \prod_i X_i \times \prod_j Y_j$

- T = complex torus;
- $X_i = X$ simply connected projective, dim $\geqslant 3$, $H^0(X, \Omega^*_X) = \mathbb{C} \oplus \mathbb{C}\omega$, where ω is a generator of K_X (**Calabi-Yau** manifolds).
- $Y_j = Y$ compact simply connected, $H^0(Y, \Omega^*_Y) = \mathbb{C}[\sigma]$, where $\sigma \in H^0(Y, \Omega^2_Y)$ is everywhere non-degenerate (**irreducible symplectic** manifolds).

Splitting the Theorem in two

To describe the history, it is convenient to split it in two theorems:

Theorem A

Let M be a compact Kähler manifold with $c_1(M) = 0$ in $H^2(M, \mathbb{R})$.
There exists $T \times X \to M$ finite étale,
T complex torus, X compact simply connected with $K_X \cong \mathscr{O}_X$.

This has highly nontrivial consequences:

Corollary

1) $K_M^{\otimes n} \cong \mathscr{O}_M$ for some n. 2) $\pi_1(M)$ is virtually abelian.

Theorem B

M compact simply connected Kähler manifold with $K_M \cong \mathscr{O}_M$
$\Longrightarrow$ $M \cong \prod_i X_i \times \prod_j Y_j$ as in the Theorem.

The Calabi conjecture

At the ICM 1954, Calabi announced (as a theorem) his now famous conjecture. In our case:

Calabi's conjecture

$c_1^{\mathbb{R}}(M) = 0 \implies M$ admits a **Ricci-flat** Kähler metric.

In a 1957 paper, he restates it as a conjecture, and gives as its main application a weak version of Theorem A:

Proposition (Calabi)

M admits a Ricci-flat Kähler metric $\Rightarrow$ Theorem A' :
$\exists\, T \times X \to M$ finite étale, T complex torus, $H^0(X, \Omega^1_X) = 0$.

By studying the automorphism group, Matsushima proved:

Proposition (Matsushima, 1969)

Theorem A' holds for M **projective** (with $c_1^{\mathbb{R}}(M) = 0$).

Bogomolov 1974

In 1974 appear 2 papers by Bogomolov:

1. *Kähler manifolds with trivial canonical class* ;
2. *On the decomposition of Kähler manifolds with trivial canonical class*.

In (1) he reproves Theorem A' in the projective case, and proves (?) $K_M^{\otimes n} \cong \mathscr{O}_M$ in the Kähler case.

In (2) he announces Theorem B (a slightly weaker form):
$K_M \cong \mathscr{O}_M$ and $\pi_1(M) = 0 \;\Rightarrow\; M \cong X \times \prod_j Y_j$,
with $H^0(X, \Omega_X^2) = 0$, Y_j symplectic.

The attempted proof of Theorem B

Sketch of proof: The heart of the proof is the following statement:

If $T_M = E \oplus F$ *with* E, F *integrable and* $\det(E) = \det(F) = \mathscr{O}_{\mathscr{M}}$, $M \cong X \times Y$ *with* $E \cong T_X$, $F \cong T_Y$.

Without the condition $\det(E) = \det(F) = \mathscr{O}_{\mathscr{M}}$, this is an open problem – there are partial results by Druel, Höring, Brunella-Pereira-Touzet. It is hard to see how the extra condition on det could help. What the paper says:

"There exists a linear connection on M *for which* E *and* F *are parallel. Hence the result"*.

The connection cannot be holomorphic (this would imply $c_i(M) = 0$ for all i). There certainly exists such a $\mathscr{C}^\infty$ connection on M (just take one on E and one on F), but then??

After Yau's theorem

In 1977 Yau announces his proof of the Calabi conjecture (the proof appears in 1978). As we will see below, the decomposition theorem is a direct consequence of Yau's theorem, plus some basic results in differential geometry.

I believe that this became soon common knowledge among differential geometers, but for some reason nobody bothered to write it down explicitely. Here is why I did it 5 years later.

In 1978 Bogomolov published another paper *Hamiltonian Kähler manifolds* where he claims that no holomorphic symplectic manifold exists in dimension > 2. The error lies in an algebraic manipulation, where I do not understand how he moves from one line to the next.

My personal involvement

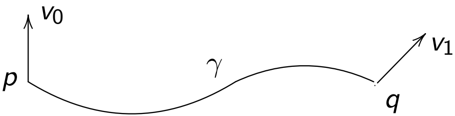
In 1982 Fujiki constructed a counter-example in dimension 4. I soon realized how to extend his construction in any dimension, then I started to study these manifolds and found a number of interesting features.

I gave a talk at Harvard beginning of 83; Phil Griffiths, who was an influential editor of the JDG at the time, suggested that I submit my paper there. He added that the JDG was looking for papers with a survey aspect, so that general remarks on manifolds with $c_1 = 0$ would be welcome. This is why I wrote a detailed proof of the decomposition theorem.

Now let me sketch how the theorem indeed follows from the Calabi conjecture.

Basics on holonomy

(M, g) Riemannian manifold $\rightsquigarrow$ parallel transport:



$\rightsquigarrow\ \varphi_\gamma : T_p(M) \xrightarrow{\sim} T_q(M)$
with $\varphi_\gamma \circ \varphi_\delta = \varphi_{\delta\gamma}$.

In particular, $\varphi : \{\text{loops at } p\} \longrightarrow \mathrm{O}(T_p(M))$;

$\operatorname{Im}\varphi := H_p =$ holonomy (sub-)group at p, closed in $\mathrm{O}(T_p(M))$.

A tensor field τ is **parallel** if $\varphi_\gamma\big(\tau(p)\big) = \tau(q)$ for every γ.

Holonomy principle

Evaluation at p gives a bijective correspondence between:

- parallel tensor fields;
- tensors on $T_p(M)$ invariant under H_p.

Examples

(M, g) with complex structure $J \in \mathrm{End}(T_M)$, $J^2 = -I$.

1. (g, J) Kähler $\iff$ J parallel $\iff$ $H_p \subset \mathrm{U}(T_p(M))$.
2. g Ricci-flat $\iff$ (K_M, g) flat $\iff$ $H_p \subset \mathrm{SU}(T_p(M))$.
3. The symplectic group:

   $$\mathrm{Sp}(r) = \mathrm{U}(2r) \cap \mathrm{Sp}(2r, \mathbb{C}) \subset \mathrm{GL}(\mathbb{C}^{2r}) = \mathrm{U}(r, \mathbb{H}) \subset \mathrm{GL}(\mathbb{H}^r)\,.$$

$H_p \subset \mathrm{Sp}(T_p(M)) \iff \exists\, \sigma$ *2-form holomorphic symplectic parallel* $\iff \exists\, I, J, K$ *parallel complex structures defining* $\mathbb{H} \to \mathrm{End}(T_M)$ (M *is* **hyperkähler**).

It is a remarkable fact that there are very few possibilities for the holonomy representation:

The de Rham and Berger theorems

From now on we assume that M is **compact** and **simply connected**.

Theorem (de Rham)

$T_p(M) = \bigoplus_i V_i$ stable under $H_p \Longrightarrow M \cong \prod_i M_i$, with $V_i = T_{p_i}(M_i)$ and $H_p \cong \prod_i H_{p_i}$.

Thus we are reduced to **irreducible** manifolds, i.e. with irreducible holonomy representation.

In his thesis (1955), Berger gave a complete list of these representations. In the special case of Kähler manifolds:

Theorem (Berger)

(M,g) Kähler non symmetric, H_p irreducible $\Rightarrow$ $H_p = \mathrm{U}, \mathrm{SU}$ or Sp.

Sketch of proof of Theorem B

Theorem B: M compact Kähler with $\pi_1(M) = 0$, $K_M = \mathscr{O}_M$.

By Yau's theorem M carries a Kähler metric which is Ricci-flat, that is, with holonomy contained in SU. By the de Rham and Berger theorems, $M \cong \prod_i X_i \times \prod_j Y_j$, where the X's have holonomy $\mathrm{SU}(n)$ and the Y's $\mathrm{Sp}(r)$ (we view $\mathrm{SU}(2)$ as $\mathrm{Sp}(1)$).

To compute $H^0(\Omega^*)$ we use the holonomy principle, plus the

Bochner principle

On a compact Kähler Ricci-flat manifold, a holomorphic tensor field is parallel.

• For $H = \mathrm{SU}(n)$, the only invariant tensor is the determinant. Thus $H^0(X, \Omega^*_X) = \mathbb{C} \oplus \mathbb{C}\omega$. Then $h^{2,0} = 0 \ \Rightarrow\ X$ projective.

• For $H = \mathrm{Sp}(r)$, the only invariant tensors are the powers of the symplectic form, hence $H^0(Y, \Omega^*_Y) = \mathbb{C}[\sigma]$.

Sketch of proof of Theorem A

M compact Kähler Ricci-flat.

Cheeger-Gromoll (1971): isometric isomorphism $\tilde{M} \xrightarrow{\sim} \mathbb{C}^k \times X$, with X compact simply connected.

Thus $M = (\mathbb{C}^k \times X)/\Gamma$, with $\Gamma \subset \mathrm{Aut}(\mathbb{C}^k) \times \mathrm{Aut}(X)$.

$\mathrm{Aut}(X)$ finite $\Rightarrow \exists\, \Gamma' \subset \Gamma$ of finite index acting trivially on X.

Bieberbach's theorem $\Rightarrow \exists\, \Gamma'' \subset \Gamma'$ of finite index acting on $\mathbb{C}^k$ by translations.

Then $(\mathbb{C}^k \times X)/\Gamma'' \cong T \times X \to M$ finite étale. ■

THE END