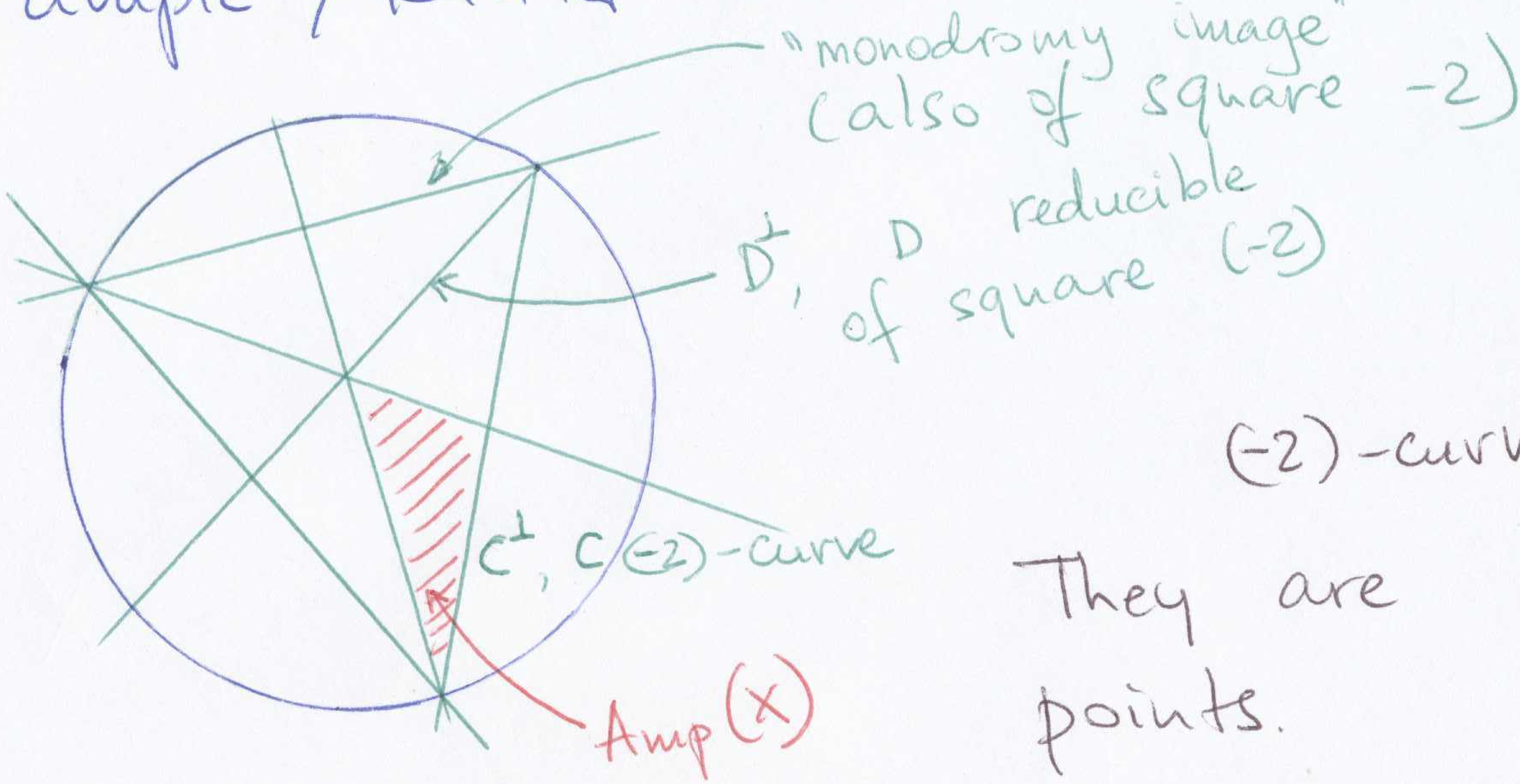
RATIONAL CURVES AND CONTRACTION LOCI ON HOLOMORPHIC SYMPLECTIC MANIFOLDS

- X projective K3-surface: a class $\alpha \in NS(X) \otimes \mathbb{R}$ is ample $\Leftrightarrow$ $\alpha^2 > 0$ and $\alpha . C > 0$ for all (-2)-curves C. (Pyatetski-Shapiro – Shafarevich). (-2)-curve: $C \subset X$, $C \simeq \mathbb{P}^1$ (then $C^2 = -2$ by adjunction)
- Same for X Kähler K3: $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ kähler $\Leftrightarrow$ $\alpha^2 > 0$, $\alpha C > 0$ $\forall C$ (-2)-curve. (Looienga-Peters).

In other words $C^\perp$, C (-2)-curve, bound the ample / kähler cone inside the "positive cone"



(-2)-curves are "extremal"

They are contracted to double points.

Higher-dimensional setting.

Def X irreducible holomorphic symplectic (IHSM): compact Kahler, $\pi_1(X) = \{e\}$, $H^{2,0}(X) = \mathbb{C}\sigma$ where σ is symplectic (= nowhere degenerate).

In particular $\dim X = 2n$ even, $K_X = \mathcal{O}_X$.

Beauville-Bogomolov form: $q: H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to \mathbb{Z}$

signature $(3, b_2-3)$, signature on $H^{1,1}_{\mathbb{R}}$: $(1, b_2-3)$

in general, not unimodular.

Example $X = \mathrm{Hilb}^n S$, S K3. $p: \mathrm{Hilb}^n S \to S^{(n)}$

Hilbert-Chow map. E except. divisor of p :

contracts onto the diagonal, double point locus

of $S^{(n)}$. $\Rightarrow [E]$ divisible by 2 in $\mathrm{Pic}(X)$.

$$H^2(X,\mathbb{Z}) \cong H^2(S,\mathbb{Z}) \oplus_{\perp} \mathbb{Z}e \qquad e = \tfrac{1}{2}[E],$$

$$q(e) = 2-2n. \qquad q|_{H^2(S,\mathbb{Z})} = \text{intersection form}$$

E is a $\mathbb{P}^1$-bundle over $\Delta \simeq S$.

Rm $H^2(S,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{Z})$ on algebraic cycles:

$[C] \longmapsto$ [subschemes with some support on C]

Rm Thanks to q, look at classes of curves as elements of $H^2(X,\mathbb{Q})$ ($H_2(X,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{Q})$ overlattice of $H^2(X,\mathbb{Z})$)

Questions • Describe the Kähler / ample cone.

Thm (Huybrechts, Boucksom) α Kähler $\Leftrightarrow$ $q(\alpha)>0$, $\alpha C>0$ $\forall$ rat. curve C. ("orthogonals to rational curves bound the Kähler cone")

Question Bound on $q(C)$? • $X=K3$: $q(C)=-2$.

• X arbitrary: bound exists, depends only on deformation type of X (A.-Verbitsky, 2015)

• X of K3 type ($X \sim_{\text{def}} \mathrm{Hilb}^n(K3)$): $q(C) \geq -\frac{n+3}{2}$ (Bayer-Hassett-Tschinkel 2015)

- Can extremal rational curves be contracted?
- Yes if X is projective: a rational class in $[C]^\perp$ is the class of a nef, big line bundle $\mathcal{L}$. $\mathcal{L}$ is semi-ample by Kawamata bpf, $\varphi_{|m\mathcal{L}|}$ contracts C.
- Also when X is arbitrary! Follows from Bakker-Lehn 2016.
- How does the exceptional set look like?

Deformation theory $\Rightarrow$ 1) A "minimal" rational curve deforms in a family of dimension $2n-2$ in X
2) It deforms outside of X together with

its cohomology class

3) If Z is a component of the corresponding exceptional locus (subvariety covered by deformations of C), then $\operatorname{codim} Z$ = relative dimension of the rational quotient (e.g. Z divisor $\Rightarrow$ Z generically a $\mathbb{P}^1$-bundle) = relative dimension of the contraction $Z \to \pi(Z)$.

Back in ± 2000: Hassett–Tschinkel conjectured a description of extremal rays and contraction loci on $\mathrm{Hilb}^2 K3$, $\mathrm{Hilb}^3 K3$, … and their deformations ("IHSM of K3 type").

<u>Example</u> $X = \mathrm{Hilb}^2 S$, S K3, $C \subset S$ (-2)-curve.

Then 3 types of rat. curves C_i, $q(C)<0$:

1) C_0 contracted by Hilbert-Chow (ruling of E)
$C_0 \sim \frac{e}{2}$, $q(C_0) = -\frac{1}{2}$.

2) C_1 line in $\mathbb{P}^2 = C^{(2)} \subset S^{[2]}$. $q(C_1) = -\frac{5}{2}$
$(C_1 \sim C - \frac{e}{2})$

3) C_2 ruling of the divisor $D_C = \{$subschemes with some support on $C\}$. $C_2 \sim C$, $q(C_2) = -2$.

Remark C_2 is not an extremal ray, but becomes so after a flop.

Hassett-Tschinkel Conjectured there are only 3 types of extremal rays, as above.

That is: $q(R) = -2$ or $-\frac{1}{2}$, exceptional set (generically) a $\mathbb{P}^1$-bundle over a K3-surface

or $q(R) = -\frac{5}{2}$, exceptional set is $\mathbb{P}^2$.

Analogous conjectures for $n>2$.

Even in low dimensions, no proof seemed to exist before the seminal work by Bayer and Macrì, who described the ample cone in terms of Mukai lattice, using stability conditions on derived categories (out of our scope here!)

I would like to sketch an easy approach in low dimensions, based on deformation to non-projective case.

PARAMETER SPACES

M underlying diff. manifold

$$\mathrm{Teich} = \mathrm{Comp}(M) / \mathrm{Diff}^0(M)$$

$\mathrm{Comp}(M)$ ← cpx structures of kähler type

$\mathrm{Diff}^0(M)$ ← isotopies

By abuse of notation: Teich connected component.

$\mathrm{Mon} \circlearrowright \mathrm{Teich}$

monodromy (part of mapping class group fixing the component)

Teich_z part where z is of type $(1,1)$ ($z \in H^2(M, \mathbb{Q})$ class of an extremal rat. curve in some complex structure)

Teich_z^+ : z positive on Kähler classes

$\cup$

Teich_z^{min} $z^\perp \supset$ wall of the Kähler cone.

"Torelli theorem" (Verbitsky)

Up to glueing unseparable points, the period map

per: Teich $\to$ $\mathbb{P}$er $\subset \mathbb{P}H^2(M,\mathbb{C})$

is an isomorphism.

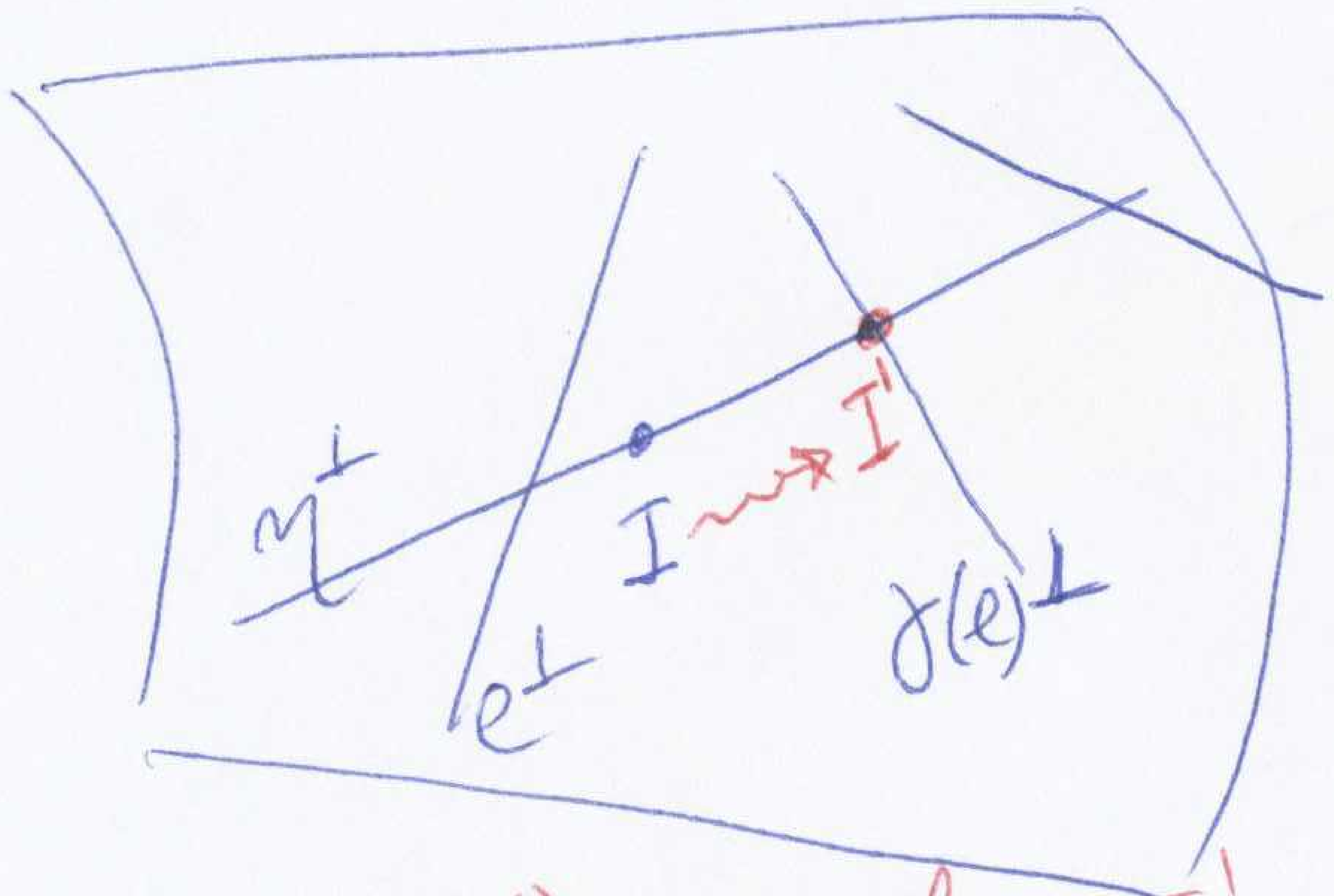
Remark. $\mathrm{Teich}_z = \mathrm{per}^{-1}(z^\perp)$

Up to inseparability, the same holds for Teich_z^+, $\mathrm{Teich}_z^{\min}$.

Thm (A.-Verbitsky '18) Let I, I' be two complex structures in $\mathrm{Teich}_z^{\min}$: $X=(M,I)$, $X'=(M,I')$ (that is, $\langle z\rangle$ is an extremal ray on both). The corresponding contraction loci are diffeomorphic, and fibers of the contraction are isomorphic whenever normal.

Now: X K3 type, Teich. The actual Hilbert schemes of K3: hyperplanes $\gamma(e)^\perp$, $\gamma \in \mathrm{Mon}$.
Let η be the class of an extremal rat. curve ($q(\eta)<0$)

Prop ($n=2$) $\exists\ \gamma \in \mathrm{Mon}$: $\langle \eta, \gamma(e)\rangle$ negative definite.
(unless η itself proportional to $\gamma(e)$).



$\rho(I') = 2$ for I' generic
($\mathrm{Pic}(X') = \langle \eta, \gamma(e)\rangle$)

Cor. X deforms to the Hilbert square of a K3 surface with cyclic Picard group, negative square of the generator.

$X' = \mathrm{Hilb}^2 S$, $\mathrm{Pic}\, S = \langle x\rangle$, $q(x) < 0$.

$\Rightarrow$ very few rational curves on X', coming from the ruling of E or from C on S (unique if exists).

Prop ($n=3,4,5$) $\nexists\ \gamma \in \mathrm{Mon} : \langle \eta, \gamma(e) \rangle$ negative semidefinite.

Cor. X deforms to $\mathrm{Hilb}^n S$, $\mathrm{Pic}\, S = \langle x \rangle$, $q(x) \leq 0$.

$q(x) < 0$ – same rat. curves as above ($n+1$ types)

$q(x) = 0$: $S \to \mathbb{P}^1$ elliptic pencil, no other curves.

get a few more types (1 if $n=3$) from linear systems on its fibers.