Testing the existence of moments for GARCH-type processes

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New Results on Time Series and their Statistical Applications

CIRM, 14-18 September, 2020

Financial time series

A standard assumption is that prices are nonstationary while returns (or log returns) are (strictly) stationary.

It is generally admitted that many financial returns series have heavy tailed marginal distributions.

However, there is no commonly accepted assumption concerning the existence of moments of such returns.

Many searchers argue that stock returns might not admit 4th-order moments (see e.g. Politis (2007)), while some even question the existence of second-order moments.

Existence of moments is central to many applications

In presence of heavy tails, many **statistical tools** developed for the analysis of financial time series become **invalid**.

For instance, using the expected shortfall in **risk analysis** requires finiteness of the first absolute moment.

Long-run horizons predictions of the squared returns require finite unconditional variance of the returns, and their confidence intervals require finite fourth-order moments.

Estimation methods may also require the existence of some moments.

Testing stationarity of financial series

Testing the existence of moments: tackled in different ways in the literature, among others

- Loretan and Phillips (1994): nonparametric methods for testing the constancy of the unconditional variance when the fourth unconditional moment is infinite.
- Estimation of the tail index for dependent observations; e.g. Hill (2015).
- Dwivedi and Subba Rao (2011): A test for second-order stationarity of a time series based on the discrete Fourier transform.
- Trapani (2016): a test for finiteness of the k-th moment of a random variable; based on the convergence/divergence of sample moments.

Testing strict stationarity in GARCH-type models:

Jensen and Rahbek (2014a, 2014b), FZ (2012, 2013), Pedersen and Rahbek (2016), Li, Zhang, Zhu and Ling (2018).

Outline

1. Tests for the standard GARCH model
   - Tests based on the Gaussian QML
   - Efficiency gains via Generalized QML
   - Numerical illustrations

2. Tests for augmented GARCH
   - Tests based on the MGF and MME
   - Power comparisons

Model: Standard GARCH(p, q)

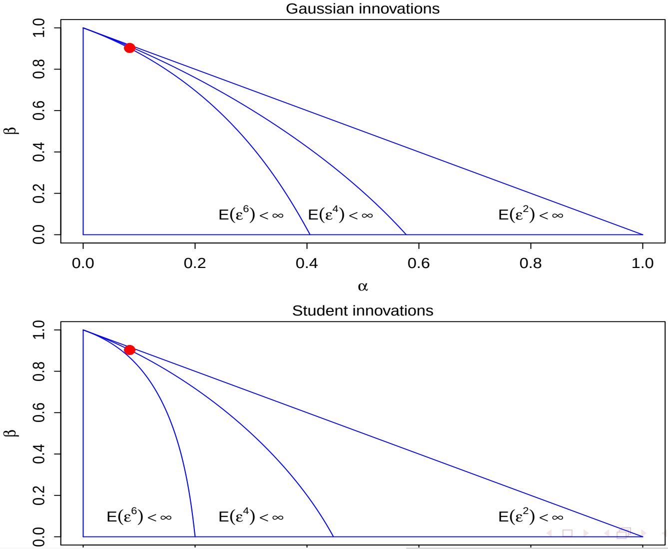
$$\begin{cases} \epsilon_t = \sigma_t \eta_t, \qquad (\eta_t) \text{ i.i.d.}, \ E\eta_t = 0, \ E\eta_t^2 = 1, \\ \sigma_t^2 = \omega_0 + \sum_{i=1}^{q} \alpha_{0i} \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_{0j} \sigma_{t-j}^2 \end{cases}$$

NSC for the existence of even-order moments depend on the moments η_t (except for the 2nd order).

[See Ling and McAleer (2002), Chen and An (1998), He and Teräsvita (1999)]

Some of these conditions are explicit (algebraic form):
2nd-order (for all p and q); $2m$-th order ($p = q = 1$)

Moment restrictions for the GARCH(1,1)



Gaussian QMLE of the GARCH(p, q)

Given observations $\epsilon_1, \dots, \epsilon_n$, and arbitrary initial values the Gaussian QMLE is defined by

$$\hat{\boldsymbol{\theta}}_n = \arg\min_{\boldsymbol{\theta} \in \Theta} \frac{1}{n} \sum_{t=1}^{n} \tilde{\ell}_t(\boldsymbol{\theta}), \quad \text{where} \quad \tilde{\ell}_t(\boldsymbol{\theta}) = \frac{\epsilon_t^2}{\tilde{\sigma}_t^2(\boldsymbol{\theta})} + \log \tilde{\sigma}_t^2(\boldsymbol{\theta}).$$

Assumptions for the CAN of the Gaussian QMLE:

A1: $\theta_0 \in \overset{\circ}{\Theta}$ and Θ is compact

A2: $\gamma(\mathbf{A}_0) < 0$, and for all $\theta \in \Theta$, $\sum_{j=1}^{p} \beta_j < 1$
[$\gamma(\mathbf{A}_0)$: top-Lyapunov exponent of the GARCH model]

A3: η_t^2 has a nondegenerate distribution and $E\eta_t^2 = 1$ and $E\eta_t^4 < \infty$

A4: The lag polynomials verify standard conditions

$$\hat{\eta}_t = \epsilon_t / \hat{\sigma}_t$$

$$\hat{\mu}_r = \frac{1}{n}\sum_{t=1}^{n} |\hat{\eta}_t|^r, \quad \mu_r = E|\eta_t|^r, \qquad \hat{\boldsymbol{\mu}}_m = (\hat{\mu}_2, \hat{\mu}_4, \ldots, \hat{\mu}_{2m})', \quad \boldsymbol{\mu}_m = (\mu_2, \mu_4, \ldots, \mu_{2m})'$$

Joint asymptotic normality of the parameter estimator and a vector of residuals sample moments

Under **A1**-**A4** and if $\mu_{4m} < \infty$

$$\begin{pmatrix} \sqrt{n}\left(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \\ \sqrt{n}(\hat{\boldsymbol{\mu}}_m - \boldsymbol{\mu}_m) \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\left\{0, \Sigma_m := \begin{pmatrix} (\mu_4 - 1)\boldsymbol{J}^{-1} & -\overline{\boldsymbol{\theta}}_0 \boldsymbol{b}_m' \\ -\boldsymbol{b}_m \overline{\boldsymbol{\theta}}_0' & \boldsymbol{A}_m \end{pmatrix}\right\},$$

$\overline{\boldsymbol{\theta}}_0 = (\omega_0, \alpha_{01}, \ldots, \alpha_{0q}, 0, \ldots, 0)'$, $\boldsymbol{J} = E\left(\frac{1}{\sigma_t^4} \frac{\partial \sigma_t^2(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}} \frac{\partial \sigma_t^2(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}'}\right)$,
$\boldsymbol{A}_m = (a_{ij})_{1 \le i,j \le m}$, $\boldsymbol{b}_m = (b_i)_{1 \le i \le m}$, with

$$\begin{aligned} a_{ij} &= \mu_{2(i+j)} + \mu_{2i}\mu_{2j}[i + j + (\mu_4 - 1)ij - 1] \\ &\quad - i\mu_{2i}\mu_{2(j+1)} - j\mu_{2j}\mu_{2(i+1)}, && 1 \le i, j \le m, \\ b_i &= \mu_{2i} - \mu_{2(i+1)} + (\mu_4 - 1)i\mu_{2i}, && 1 \le i \le m. \end{aligned}$$

Remarks

1. The asymptotic variance-covariance matrix $\boldsymbol{A}_m$ of the vector of empirical moments of the rescaled returns is **model free** (does not depend on $\boldsymbol{\theta}_0$) but **not estimation free**.
   This is due to the relation

$$\boldsymbol{\Omega}'\boldsymbol{J}^{-1}\boldsymbol{\Omega} = 1$$

$\boldsymbol{\Omega} = E\left(\frac{1}{\sigma_t^2}\frac{\partial\sigma_t^2(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\right)$, $\boldsymbol{J} = E\left(\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\frac{\partial\sigma_t^2(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'}\right)$ (see FZ (2013)).

2. **Case $m = 1$ degenerate**

   $\hat{\boldsymbol{\mu}}_1 = 1$ whence the initial values are such that

   for any $K > 0$, $K\tilde{\sigma}_t^2(\hat{\boldsymbol{\theta}}_n) = \tilde{\sigma}_t^2(\hat{\boldsymbol{\theta}}_n^*)$ for some $\hat{\boldsymbol{\theta}}_n^* \in \Theta$.

   For more general initial values,

$$\sqrt{n}(\hat{\boldsymbol{\mu}}_2 - 1) \to 0, \quad \text{in probability as } n \to \infty.$$

GARCH(1,1) case: $\sigma_t^2 = \omega_0 + \alpha_0 \epsilon_{t-1}^2 + \beta_0 \sigma_{t-1}^2$

If $m \geq 1$ is an integer,

$$E(\epsilon_t^{2m}) < \infty \quad \Leftrightarrow \quad \sum_{i=0}^{m} \binom{m}{i} \alpha_0^i \beta_0^{m-i} \mu_{2i} < 1$$

Let $G(\boldsymbol{\theta}, \boldsymbol{\mu}) = \sum_{i=0}^{m} \binom{m}{i} \alpha^i \beta^{m-i} \mu_{2i}$. Under the previous assumptions

$$\sqrt{n}\{G(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\mu}}_m) - G(\boldsymbol{\theta}_0, \boldsymbol{\mu}_m)\} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_m^2),$$

where

$$\sigma_m^2 = \frac{\partial G(\boldsymbol{\theta}_0, \boldsymbol{\mu}_m)}{\partial(\boldsymbol{\theta}', \boldsymbol{\mu}')} \boldsymbol{\Sigma}_m \frac{\partial G(\boldsymbol{\theta}_0, \boldsymbol{\mu}_m)}{\partial \binom{\boldsymbol{\theta}}{\boldsymbol{\mu}}}.$$

Testing problems

Consider the $2m$-th order stationarity problems

$$\boldsymbol{H}_0: \quad E(\epsilon_t^{2m}) < \infty \quad \text{against} \quad \boldsymbol{H}_1: \quad E(\epsilon_t^{2m}) = \infty,$$

and

$$\boldsymbol{H}_0^*: \quad E(\epsilon_t^{2m}) = \infty \quad \text{against} \quad \boldsymbol{H}_1^*: \quad E(\epsilon_t^{2m}) < \infty.$$

Test of $2m$-th order moment for the GARCH(1,1)

Test of H_0 (resp. H_0^*) at the asymptotic level $\alpha \in (0,1)$

Defined by the rejection region

$$\{T_n > \Phi^{-1}(1-\alpha)\}, \qquad \text{(resp. } \{T_n < \Phi^{-1}(\alpha)\}\text{)},$$

where

$$T_n = \frac{\sqrt{n}\left\{\sum_{i=0}^{m}\binom{m}{i}\hat{\alpha}_n^i\hat{\beta}_n^{m-i}\hat{\mu}_{2i} - 1\right\}}{\hat{\sigma}_m},$$

$$\hat{\sigma}_m^2 = \frac{\partial G(\hat{\boldsymbol{\theta}}_n, \hat{\boldsymbol{\mu}}_m)}{\partial(\boldsymbol{\theta}', \boldsymbol{\mu}')}\hat{\boldsymbol{\Sigma}}_m\frac{\partial G(\hat{\boldsymbol{\theta}}_n, \hat{\boldsymbol{\mu}}_m)}{\partial\binom{\boldsymbol{\theta}}{\boldsymbol{\mu}}}$$

and $\hat{\boldsymbol{\Sigma}}_m$ is a consistent estimator of $\boldsymbol{\Sigma}_m$.

Remarks

- The test is constructed for the closure of the null assumption
  $\overline{\boldsymbol{H}}_0: \quad \sum_{i=0}^{m} \binom{m}{i} \alpha_0^i \beta_0^{m-i} \mu_{2i} \leq 1.$

  The asymptotic region satisfies

$$\sup_{\overline{\boldsymbol{H}}_0} \lim_{n\to\infty} P\{T_n > \Phi^{-1}(1-\alpha)\} = \alpha$$

- Testing the 2nd-order moment condition: $\alpha_0 + \beta_0 < 1$
  In this case, with $\boldsymbol{e} = (0, 1, 1)'$,

$$T_n = \frac{\sqrt{n}(\hat{\alpha} + \hat{\beta} - 1)}{\{(\hat{\mu}_4 - 1)\boldsymbol{e}'\hat{\boldsymbol{J}}^{-1}\boldsymbol{e}\}^{1/2}}.$$

- A bootstrap procedure can be used to avoid estimating the asymptotic distribution.

Resampling scheme

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Reparametrization

Provided that $E|\eta_t|^r < \infty$, the GARCH(p,q) model can be equivalently rewritten as

$$\epsilon_t = \sigma_t(\boldsymbol{\theta}_0^{(r)})\eta_t^{(r)}, \qquad E|\eta_t^{(r)}|^r = 1,$$

where $\eta_t^{(r)} = \eta_t/\{E|\eta_t|^r\}^{1/r}$.

Link with the original parameters:

$$\boldsymbol{\theta}_0 = B^{(r)}\boldsymbol{\theta}_0^{(r)}, \quad B^{(r)} = \begin{pmatrix} \mu_r^{-2/r} I_{q+1} & 0 \\ 0 & I_p \end{pmatrix} = \begin{pmatrix} \mu_2^{(r)} I_{q+1} & 0 \\ 0 & I_p \end{pmatrix},$$

where $\mu_s^{(r)} = E|\eta_t^{(r)}|^s$ for any $s > 0$.

Generalized QMLE of $\boldsymbol{\theta}_0^{(r)}$

For $\Theta^{(r)}$ such that $\Theta = \{B^{(r)}\boldsymbol{\theta}, \quad \boldsymbol{\theta} \in \Theta^{(r)}\}$

$$\widehat{\boldsymbol{\theta}}_n^{(r)} = \underset{\boldsymbol{\theta}\in\Theta^{(r)}}{\operatorname{argmin}}\ \tilde{\mathbf{I}}_n(\boldsymbol{\theta}),$$

where for $\boldsymbol{\theta} \in \Theta^{(r)}$,

$$\tilde{\mathbf{I}}_n(\boldsymbol{\theta}) = \frac{1}{n}\sum_{t=1}^{n}\tilde{l}_t(\boldsymbol{\theta}) \quad \text{with} \quad \tilde{l}_t(\boldsymbol{\theta}) = \log\tilde{\sigma}_t^2(\boldsymbol{\theta}) + \frac{2}{r}\frac{|\epsilon_t|^r}{\tilde{\sigma}_t^r(\boldsymbol{\theta})}.$$

Remark: under the identifiability constraint $E|\eta_t^{(r)}|^r = 1$, the only QMLE which is strongly consistent (to $\boldsymbol{\theta}_0^{(r)}$, not to $\boldsymbol{\theta}_0$) whatever the error distribution is of the above form (cf FZ, 2013).

Two-stage QMLE

$$\boldsymbol{\theta}_0 = B^{(r)}\boldsymbol{\theta}_0^{(r)}$$

$B^{(r)}$ can be estimated from empirical moments of the standardized returns $\widehat{\eta}_t^{(r)} = \epsilon_t/\tilde{\sigma}_t(\widehat{\boldsymbol{\theta}}_n^{(r)})$.

Asymptotic law of the *two-stage* QMLE of $\boldsymbol{\theta}_0$ (FLZ, 2011)

Let $r > 0$. Under Assumptions **A1**-**A4**, and if $\mu_{2r} < \infty$,

$$\sqrt{n}\left(\widehat{B}_n^{(r)}\widehat{\boldsymbol{\theta}}_n^{(r)} - \boldsymbol{\theta}_0\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \Sigma^{(r)}\right),$$

$$\Sigma^{(r)} = g(r)J^{-1} + \{\mu_4 - 1 - g(r)\}\,\overline{\boldsymbol{\theta}}_0\overline{\boldsymbol{\theta}}_0',$$

$$g(r) = \left(\frac{2}{r}\right)^2\left(\frac{\mu_{2r}}{\mu_r^2} - 1\right), \quad \overline{\boldsymbol{\theta}}_0 = (\omega_0, \alpha_{01}, \dots, \alpha_{0q}, 0, \dots, 0)'$$

[For the Gaussian QML ($r = 2$) we have $\Sigma^{(2)} = (\mu_4 - 1)J^{-1}$.]

Testing second-order stationarity using the 2QMLE of $\boldsymbol{\theta}_0$

Let

$$\mathbf{H_0}: \quad \sum_{i=1}^{q} \alpha_{0i} + \sum_{j=1}^{p} \beta_{0j} < 1, \quad \text{or, equivalently} \quad \mathbf{H_0}: \quad \boldsymbol{c}'\boldsymbol{\theta}_0 < 1,$$

where $\boldsymbol{c} = (0, 1, \ldots, 1) \in \mathbb{R}^{p+q+1}$.

Test of $\boldsymbol{H}_0$ [resp. $\boldsymbol{H}_0^*: \ \boldsymbol{c}'\boldsymbol{\theta}_0 \geq 1$] at level $\alpha \in (0, 1)$

Defined by the rejection region

$$\mathrm{C}_r = \{T_{n,r} > \Phi^{-1}(1-\alpha)\}, \quad [\text{resp. } \mathrm{C}_r^* = \{T_{n,r} < \Phi^{-1}(\alpha)\}].$$

where

$$T_{n,r} = \frac{\sqrt{n}(\boldsymbol{c}'\widehat{B}_n^{(r)}\widehat{\boldsymbol{\theta}}_n^{(r)} - 1)}{\boldsymbol{c}'\widehat{\Sigma}^{(r)}\boldsymbol{c}}.$$

Local alternatives

Around $\boldsymbol{\theta}_0$ such that $\boldsymbol{c}'\boldsymbol{\theta}_0 = 1$, let a sequence of local parameters

$$\boldsymbol{\theta}_n = \boldsymbol{\theta}_0 + \frac{\boldsymbol{\tau}}{\sqrt{n}}, \quad \boldsymbol{\tau} \in \mathbb{R}^{p+q+1}.$$

Regularity assumptions on the density f of η_t:

$$f > 0, \qquad \lim_{|y|\to\infty} yf(y) = 0, \qquad \lim_{|y|\to\infty} y^2 f'(y) = 0,$$

and for $K > 0$ and $\delta > 0$,

$$|y| \left| \frac{f'}{f}(y) \right| + y^2 \left| \left(\frac{f'}{f} \right)' (y) \right| + y^2 \left| \left(\frac{f'}{f} \right)'' (y) \right| \le K \left(1 + |y|^\delta \right),$$

$$E\,|\eta_1|^{2\delta} < \infty.$$

Local alternatives

Local Asymptotic Powers

Local asymptotic powers of the 2nd-order stationarity tests:

$$\lim_{n\to\infty} P_{n,\boldsymbol{\tau}}\left(\mathrm{C}_r\right) = \Phi\left\{\Phi^{-1}(\alpha) + \frac{\boldsymbol{c}'\boldsymbol{\tau}}{\sigma^{(r)}}\right\} \quad \text{for} \quad \boldsymbol{c}'\boldsymbol{\tau} \geq 0,$$

$$\lim_{n\to\infty} P_{n,\boldsymbol{\tau}}\left(\mathrm{C}_r^*\right) = \Phi\left\{\Phi^{-1}(\alpha) - \frac{\boldsymbol{c}'\boldsymbol{\tau}}{\sigma^{(r)}}\right\} \quad \text{for} \quad \boldsymbol{c}'\boldsymbol{\tau} \leq 0.$$

Comparison when r varies thus boils down to comparing the coefficients

$$\sigma^{(r)} = \left\{\boldsymbol{c}'\Sigma^{(r)}\boldsymbol{c}\right\}^{1/2}.$$

Local comparisons

Optimal r

Let $[\underline{r}, \overline{r}]$ such that r_0 is well defined, where

$$r_0 = \arg\min_{[\underline{r}, \overline{r}]} g(r), \qquad g(r) = \left(\frac{2}{r}\right)^2 \left(\frac{\mu_{2r}}{\mu_r^2} - 1\right).$$

Then, within the family $\{C_r, r \in [\underline{r}, \overline{r}]\}$ for testing $\mathbf{H_0}$ the test C_{r_0} has the highest local asymptotic power, uniformly in $\boldsymbol{\tau}$.

Remarks:

1. r_0 depends on the errors distribution, and is also optimal for the estimator $\widehat{\boldsymbol{\theta}}_{n,r}$ of θ_0.
   If $\eta_t \sim \mathcal{N}(0,1)$, $r_0 = 2$, but in general tests based on the GQMLE are not optimal. If $\eta_t \sim t(\nu)$: $r_0 < 1$ for small values of ν, and increases to 2 as $\nu \to \infty$.
2. A minimum of g over $\mathbb{R}^+$ may not exist for particular distributions of η_t.
3. r_0 is not known but can be consistently estimated.

Performance of tests of existence of $2m$th-order moments

$N = 1000$ independent trajectories of size $n = 2000,\ 4000,\ 8000$ of a GARCH(1,1):

$$\begin{cases} \epsilon_t = \sigma_t \eta_t, & (\eta_t) \text{ i.i.d.} \mathcal{N}(0,1) \\ \sigma_t^2 = 0.5 + 0.105\epsilon_{t-1}^2 + 0.87\sigma_{t-1}^2 \end{cases}$$

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
$\sum_{i=0}^{m} \binom{m}{i} \alpha_0^i \beta_0^{m-i} \mu_{2i} - 1$	-0.025	-0.027	0.001	0.073	0.216	0.482

Thus, for integers m,

$$E|\epsilon_t|^{2m} < \infty \quad \Leftrightarrow \quad m \leq 2$$

Finite-sample performance

Table: Relative frequency of rejection of $H_0:\ E\epsilon_t^{2m}<\infty$ against $H_1:\ E\epsilon_t^{2m}=\infty$ [or of $H_0^*:\ E\epsilon_t^{2m}=\infty$ against $H_1^*:\ E\epsilon_t^{2m}<\infty$] at the nominal level 5% or 10%.

Null	n	level	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
H_0	2000	5%	0.0	0.0	1.2	14.4	35.8	48.9
		10%	0.0	0.0	4.5	30.6	60.5	80.6
	4000	5%	0.0	0.0	2.4	35.9	77.1	93.1
		10%	0.0	0.0	6.4	53.4	90.0	98.5
	8000	5%	0.0	0.0	3.0	66.8	99.0	99.9
		10%	0.0	0.0	6.9	79.6	99.6	100.0
H_0^*	2000	5%	97.5	48.1	7.9	0.7	0.1	0.1
		10%	99.8	65.9	15.7	1.8	0.1	0.1
	4000	5%	100.0	72.7	7.3	0.1	0.0	0.0
		10%	100.0	85.3	14.7	0.4	0.0	0.0
	8000	5%	100.0	94.1	6.7	0.0	0.0	0.0
		10%	100.0	97.3	14.5	0.0	0.0	0.0

Using Bootstrap

Table: Using the resampling algorithm instead of the asymptotic distribution

Null	n	level	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
H_0	2000	5%	0.0	0.1	3.6	24.8	50.2	72.9
		10%	0.0	0.1	8.3	38.4	67.6	86.8
	4000	5%	0.0	0.0	6.3	42.9	81.5	94.7
		10%	0.0	0.1	11.0	60.2	89.7	98.6
	8000	5%	0.0	0.0	4.3	68.3	97.9	99.8
		10%	0.0	0.0	9.1	81.5	99.4	100.0
H_0^*	2000	5%	83.3	31.2	4.3	0.6	0.0	0.0
		10%	95.1	48.9	9.7	1.3	0.1	0.0
	4000	5%	98.9	51.9	4.5	0.1	0.0	0.0
		10%	100.0	69.8	10.2	0.7	0.0	0.0
	8000	5%	100.0	81.8	5.3	0.0	0.0	0.0
		10%	100.0	93.3	10.2	0.1	0.0	0.0

Empirical distribution of $S_n = \hat{\alpha} + \hat{\beta}$ when $\alpha + \beta = 1$.

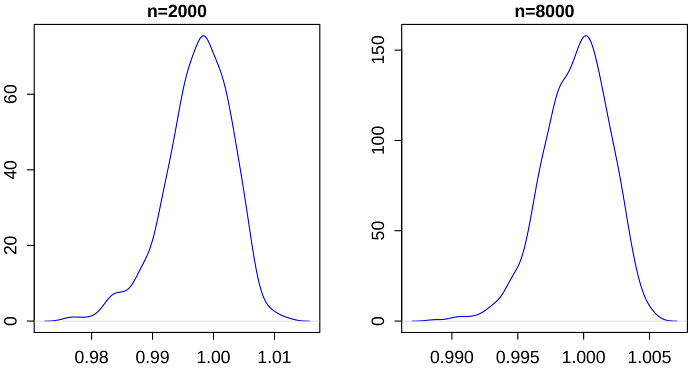


Figure: Based on 1,000 simulations of a GARCH(1,1) with $\alpha = 0.1$, $\beta = 0.9$ and $\eta_t \sim \mathcal{N}(0, 1)$.

Tests based on non-Gaussian QML

Let the empirical function

$$r \mapsto \hat{g}(r) = \left(\frac{2}{r}\right)^2 \left(\frac{\hat{\mu}_{2r}}{\hat{\mu}_r^2}\right)$$

for $r \in [\underline{r}, \overline{r}]$ when $\eta_t \sim \mathcal{N}(0,1)$.

The optimal value $r_0 = 2$ of r minimizes the theoretical function $g(r)$.

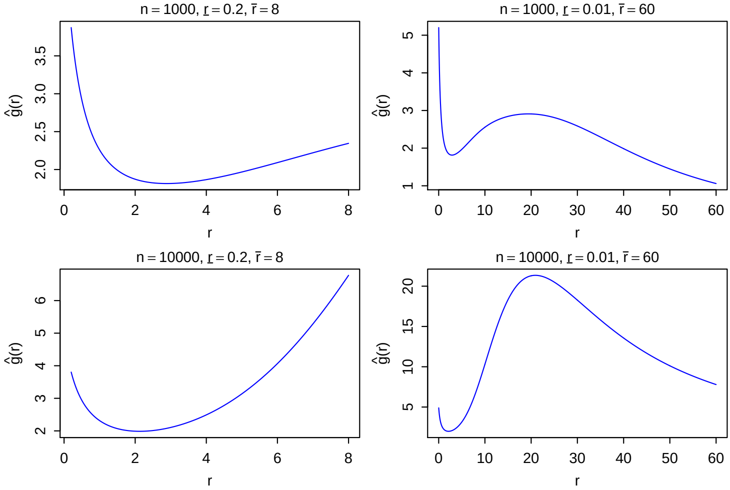


Figure: Empirical estimate of the function $g(r)$ when the GARCH innovation $\eta_t \sim \mathcal{N}(0,1)$.

Finite-sample performance for non-Gaussian errors

Table: Relative frequency of rejection of $H_0: \ E\epsilon_t^2 < \infty$ against $H_1: \ E\epsilon_t^2 = \infty$ or of $H_0^*: \ E\epsilon_t^2 = \infty$ against $H_1^*: \ E\epsilon_t^2 < \infty$ at the nominal level 5% or 10%, using the Gaussian QML or the generalized QML methods.
$\eta_t \sim$ GED(0.3)

(α_0, β_0)			$(0.1, 0.8)$		$(0.105, 0.87)$		$(0.105, 0.895)$		$(0.145, 0.88)$	
$\alpha_0 + \beta_0$			0.9		0.975		1		1.025	
Null	n	level	QML	gQML	QML	gQML	QML	gQML	QML	gQML
H_0	2000	5%	0.0	0.0	0.2	0.0	0.4	1.8	2.6	8.9
		10%	0.2	0.0	0.8	0.4	2.8	5.3	9.7	22.4
	4000	5%	0.0	0.0	0.1	0.0	1.2	1.6	6.3	21.0
		10%	0.0	0.0	0.8	0.2	4.5	6.3	19.3	37.1
	8000	5%	0.0	0.0	0.2	0.0	2.1	3.1	14.8	43.3
		10%	0.1	0.0	0.8	0.1	6.2	7.8	31.2	61.6
H_0^*	2000	5%	6.5	84.4	2.2	33.5	0.7	10.4	0.5	2.9
		10%	25.6	91.1	16.6	47.4	6.8	15.6	4.2	5.0
	4000	5%	35.1	98.4	13.7	44.1	5.5	10.1	1.3	1.3
		10%	69.8	98.7	35.6	56.1	17.5	16.2	4.3	1.9
	8000	5%	87.2	100.0	31.3	58.7	8.2	7.6	1.4	0.2
		10%	94.6	100.0	46.5	69.0	15.4	13.3	2.5	0.9

Finite-sample performance with bootstrap

Table: Using resampling algorithms instead of the asymptotic distributions.

(α_0, β_0)			$(0.1, 0.8)$		$(0.105, 0.895)$		$(0.15, 0.9)$	
$\alpha_0 + \beta_0$			0.9		1		1.05	
Null	n	α	QML	gQML	QML	gQML	QML	gQML
H_0	2000	5%	0.3	0.0	2.7	4.3	21.0	41.0
		10%	1.0	0.1	6.7	8.8	40.0	59.6
H_0^*	2000	5%	14.2	31.9	3.6	3.1	0.2	0.5
		10%	30.7	51.1	8.1	7.1	0.7	0.6

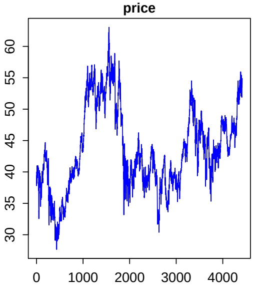
Empirical appplication

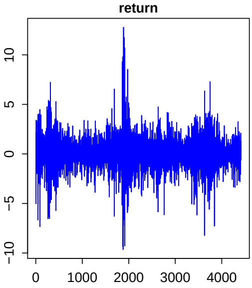
Daily stock returns of Total SA (2001-07-16 to 2018-09-21)

Estimated GARCH(1,1) model:

$$\hat{\omega} = 0.035(0.009), \quad \hat{\alpha} = 0.083(0.011), \quad \hat{\beta} = 0.903(0.011)$$

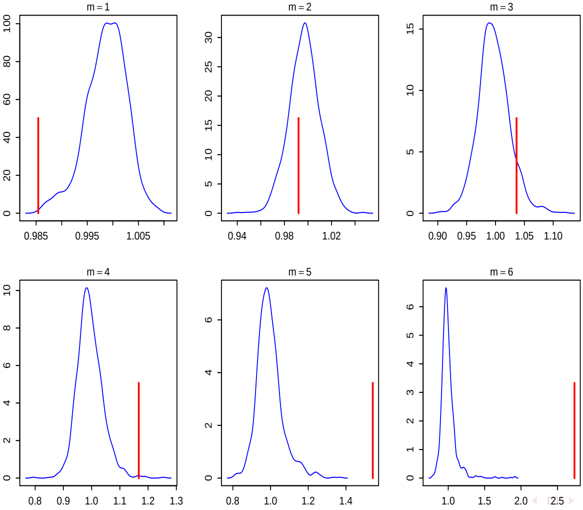
	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
T_n	-2.96	-0.69	1.15	1.62	1.45	1.19





Estimator of the density of S_n under the null that $S = 1$

Value of $S_n = \sum_{i=0}^{m} \binom{m}{i} \hat{\alpha}^i \hat{\beta}^{m-i} \hat{\mu}_{2i}$ computed from the observations = vertical line



Augmented GARCH processes

Many GARCH(1,1)-type models are of the form

$$\epsilon_t = \sigma_t \eta_t, \qquad \sigma_t^\delta = \omega(\eta_{t-1}) + a(\eta_{t-1})\sigma_{t-1}^\delta,$$

$\delta > 0$, $\omega : \mathbb{R} \to [\underline{\omega}, +\infty)$ and $a : \mathbb{R} \to [\underline{a}, +\infty)$, for some $\underline{\omega} > 0$ and $\underline{a} \geq 0$.

[Duan (1997), He and Teräsvirta (1999), Aue, Berkes and Horváth (2006)]

Example [standard GARCH]: $\omega(\eta) = \omega$, $a(\eta) = \alpha\eta^2 + \beta$ and $\delta = 2$.

- **Strict stationarity condition:** $\gamma = \mathbf{E} \log \mathbf{a}(\eta_1) < \mathbf{0}$
  (assuming $E \log^+ a(\eta_1) < \infty$ and $E \log^+ \omega(\eta_1) < \infty$)
- **Existence of moments condition:** for $u > 0$,
  $E(\sigma_t^{u\delta}) < \infty \quad \Leftrightarrow \quad \mathbf{E}[\mathbf{a}^\mathbf{u}(\eta_1)] < \mathbf{1}$ and $E[\omega^u(\eta_1)] < \infty$.

$u \mapsto E[a^u(\eta_1)]$ can be called Moment Generating Function (MGF) of the augmented GARCH model.

Augmented GARCH models

for $\delta_0 > 0$ and $\boldsymbol{\theta}_0 \in \Theta \subset \mathbb{R}^d$,

$$\epsilon_t = \sigma_t(\boldsymbol{\theta}_0)\eta_t, \quad \sigma_t^{\delta_0}(\boldsymbol{\theta}_0) = \omega(\eta_{t-1}; \boldsymbol{\theta}_0) + a(\eta_{t-1}; \boldsymbol{\theta}_0)\sigma_{t-1}^{\delta_0}$$

For any $\boldsymbol{\theta} \in \Theta$, $\omega(\cdot; \boldsymbol{\theta}) : \mathbb{R} \to [\underline{\omega}, +\infty)$ and $a(\cdot; \boldsymbol{\theta}) : \mathbb{R} \to [\underline{a}, +\infty)$.

(ϵ_t): **strictly stationary, non-anticipative and ergodic solution**

Given observations $\epsilon_1, \ldots, \epsilon_n$, and arbitrary **initial values** $\tilde{\epsilon}_0$ and $\tilde{\sigma}_0 > 0$ let, for $t = 1, \ldots, n$ and any $\boldsymbol{\theta} \in \Theta$,

$$\tilde{\sigma}_t^{\delta}(\boldsymbol{\theta}) = \omega\left(\frac{\epsilon_{t-1}}{\tilde{\sigma}_{t-1}(\boldsymbol{\theta})}; \boldsymbol{\theta}\right) + a\left(\frac{\epsilon_{t-1}}{\tilde{\sigma}_{t-1}(\boldsymbol{\theta})}; \boldsymbol{\theta}\right)\tilde{\sigma}_{t-1}^{\delta}$$

SRE:

$$\sigma_t^{\delta}(\boldsymbol{\theta}) = \omega\left(\frac{\epsilon_{t-1}}{\sigma_{t-1}(\boldsymbol{\theta})}; \boldsymbol{\theta}\right) + a\left(\frac{\epsilon_{t-1}}{\sigma_{t-1}(\boldsymbol{\theta})}; \boldsymbol{\theta}\right)\sigma_{t-1}^{\delta}, \qquad t \in \mathbb{Z}.$$

Main assumptions

B1: [Strict stationarity] $E \log^+ \omega(\eta_1, \boldsymbol{\theta}_0) < \infty$, $E \log a(\eta_1, \boldsymbol{\theta}_0) < 0$ and $E[a^s(\eta_1, \boldsymbol{\theta}_0)] < \infty$ for some $s > 0$.

B2: [Existence of a solution to the SRE] For any $\boldsymbol{\theta} \in \Theta$, there exists $z_0 > 0$ such that

$$E \log^+ \omega\left(\frac{\epsilon_t}{z_0^{1/\delta}}; \boldsymbol{\theta}\right) + \log^+ a\left(\frac{\epsilon_t}{z_0^{1/\delta}}; \boldsymbol{\theta}\right) < \infty,$$

$$E \log \sup_{z \geq \underline{\omega}} \left| \frac{\partial}{\partial z} \left\{ \omega\left(\frac{\epsilon_t}{z^{1/\delta}}; \boldsymbol{\theta}\right) + a\left(\frac{\epsilon_t}{z^{1/\delta}}; \boldsymbol{\theta}\right) z \right\} \right| < 0.$$

B3: [Invertibility] The $\mathcal{F}_{t-1}$-measurable function $\boldsymbol{\theta} \to (\sigma_t(\boldsymbol{\theta}), \tilde{\sigma}_t(\boldsymbol{\theta}))$ is a.s. $\mathcal{C}^1$.

$$\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} |\sigma_t(\boldsymbol{\theta}) - \tilde{\sigma}_t(\boldsymbol{\theta})| + \left| \frac{\partial \sigma_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} - \frac{\partial \tilde{\sigma}_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right| \leq K_t \rho^t$$

where $K_t \in \mathcal{F}_{t-1}$ and $\sup_t E(K_t^r) < \infty$ for some $r > 0$.

B4: [Bahadur expansion]

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \boldsymbol{\Delta}_{t-1} \boldsymbol{V}(\eta_t) + o_P(1), \qquad \boldsymbol{\Delta}_{t-1} \in \mathcal{F}_{t-1}$$

with $E\boldsymbol{V}(\eta_t) = 0$, var$\{\boldsymbol{V}(\eta_t)\} = \boldsymbol{\Upsilon}$ is nonsingular, $E\boldsymbol{\Delta}_t = \boldsymbol{\Lambda}$ is full row rank.

Asymptotic distribution of the empirical MGF

Recalling the **moment condition** $E[a^u(\eta_1)] < 1$, a test statistic can be based on the empirical MGF: $S_n^{(u)} = \frac{1}{n}\sum_{t=1}^n a^u(\hat{\eta}_t; \widehat{\boldsymbol{\theta}}_n)$

Let $S_\infty^{(u)} = E[a^u(\eta_t; \boldsymbol{\theta}_0)]$.

Asymptotic distribution of $S_n^{(u)}$

For $0 < u \leq s/2$, we have

$$\sqrt{n}\left\{S_n^{(u)} - S_\infty^{(u)}\right\} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, v_u^2 := \boldsymbol{g}_u'\boldsymbol{\Sigma}\boldsymbol{g}_u + \psi_u + 2\boldsymbol{g}_u'\boldsymbol{\xi}_u\right).$$

where $\boldsymbol{\Sigma} = E(\boldsymbol{\Delta}_t \boldsymbol{\Upsilon} \boldsymbol{\Delta}_t')$, $\psi_u = \mathrm{Var}[a^u(\eta_1; \boldsymbol{\theta}_0)]$, $\boldsymbol{\xi}_u = \boldsymbol{\Lambda} E[\boldsymbol{V}(\eta_t) a^u(\eta_t; \boldsymbol{\theta}_0)]$,

$$\boldsymbol{g}_u = E\left(\boldsymbol{g}_{u,t}\right) \quad \text{where} \quad \boldsymbol{g}_{u,t} = \left[\frac{\partial}{\partial \boldsymbol{\theta}} a^u\{\eta_t(\boldsymbol{\theta}); \boldsymbol{\theta}\}\right]_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}.$$

Moreover $v_u^2 > 0$ if $\mathrm{Var}\left\{a^u(\eta_t; \boldsymbol{\theta}_0), \boldsymbol{V}'(\eta_t)\right\}$ is positive definite.

Particular cases

QML and ML for the standard GARCH(1,1)

$M_{x,y} = E[\eta_t^{2x}(\alpha_0\eta_t^2 + \beta_0)^y]$, $x, y \in \mathbb{R}$, $\boldsymbol{m}_u = (0, M_{1,u-1}, M_{0,u-1})'$,

$$\boldsymbol{J} = E\left(\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\frac{\partial\sigma_t^2(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'}\right).$$

Asymptotic variances of $S_n^{(u)} = \frac{1}{n}\sum_{t=1}^n a^u(\hat{\eta}_t; \widehat{\boldsymbol{\theta}}_n)$:

$$\begin{aligned} \upsilon^2_{u,QML} &= u^2(\kappa_4 - 1)\left\{\boldsymbol{m}_u'\boldsymbol{J}^{-1}\boldsymbol{m}_u - \alpha_0^2 M^2_{1,u-1}\right\} + M_{0,2u} - M^2_{0,u}, \\ \upsilon^2_{u,ML} &= = \frac{4u^2}{\iota_f}\left\{\boldsymbol{m}_u'\boldsymbol{J}^{-1}\boldsymbol{m}_u - \alpha_0^2 M^2_{1,u-1}\right\} + M_{0,2u} - M^2_{0,u} \end{aligned}$$

where $\kappa_4 = E\eta_t^4$ and $\iota_f = \int\{1 + yf'(y)/f(y)\}^2 f(y)dy$ is the Fisher information for scale.

Testing the existence of moments of given order

$$\boldsymbol{H}_{0,u}: \quad E\{a^u(\eta_t)\} < 1 \quad \text{against} \quad \boldsymbol{H}_{1,u}: \quad E\{a^u(\eta_t)\} \geq 1$$

$$\boldsymbol{H}^*_{0,u}: \quad E\{a^u(\eta_t)\} \geq 1 \quad \text{against} \quad \boldsymbol{H}^*_{1,u}: \quad E\{a^u(\eta_t)\} < 1$$

Test statistic based on the empirical MGF

$$T_n^{(u)} = \frac{\sqrt{n}\left\{S_n^{(u)} - 1\right\}}{\hat{\upsilon}_u}, \quad \text{where} \quad \hat{\upsilon}_u^2 = \hat{\boldsymbol{g}}_u' \hat{\boldsymbol{\Sigma}} \hat{\boldsymbol{g}}_u + \hat{\psi}_u + 2\hat{\boldsymbol{g}}_u' \hat{\xi}_u,$$

Test of $\boldsymbol{H}_{0,u}$ [resp. $\boldsymbol{H}^*_{0,u}$] at the asymptotic level $\underline{\alpha} \in (0,1)$

$$C_T^{(u)} = \{T_n^{(u)} > \Phi^{-1}(1-\underline{\alpha})\}, \qquad [\text{resp. } \{T_n^{(u)} < \Phi^{-1}(\underline{\alpha})\}],$$

Maximal Moment exponent

Augmented GARCH:

$$\epsilon_t = \sigma_t \eta_t, \quad \sigma_t^\delta = \omega(\eta_{t-1}) + a(\eta_{t-1})\sigma_{t-1}^\delta$$

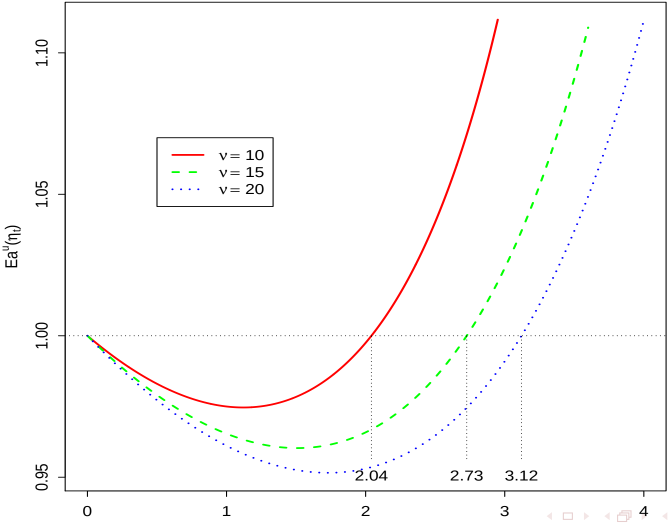
The **Maximal Moment Exponent** (MME), when existing, is the maximal order u_0 at which moments of σ_t^δ exist:

$$\begin{aligned} u_0 &= \sup\{u > 0; \quad E\sigma_t^{\delta u} < \infty\} \\ &= \sup\{u > 0; \quad E\{a^u(\eta_t)\} < 1\}, \end{aligned}$$

assuming $E\omega^u(\eta_t) < \infty$ for all $u > 0$.

Berkes, Horváth and Kokoszka (2003) proposed an estimator of this coefficient for standard GARCH(1,1) models.

MGF and MME for a GARCH(1,1) and Student distributions



Condition for a finite MME

Suppose $\gamma = E\log a(\eta_1) < 0$

- If $P[a(\eta_1) \leq 1] = 1$, then $\forall u > 0$, $E[a^u(\eta_1)] < 1$, and $E(\sigma_t^{u\delta}) < \infty$ provided $E[\omega^u(\eta_1)] < \infty$. We set $u_0 = \infty$.
- If $P[a(\eta_1) \leq 1] < 1$, and $1 \leq E[a^s(\eta_1)] < \infty$ for some $s > 0$,

  there exists a unique $u_0 > 0$ such that $\mathbf{E[a^{u_0}(\eta_1)] = 1}$.

If $E[\omega^{u_0}(\eta_1)] < \infty$,

$$E(\sigma_t^{u\delta}) < \infty, \quad \forall u < u_0,$$
$$E(\sigma_t^{u\delta}) = \infty, \quad u \geq u_0.$$

Empirical MME

Suppose $\gamma_n := \frac{1}{n}\sum_{t=1}^{n} \log a(\hat{\eta}_t; \widehat{\boldsymbol{\theta}}_n) < 0$

- If $a(\hat{\eta}_t; \widehat{\boldsymbol{\theta}}_n) \leq 1$ for $t = 1, \ldots, n$, then $S_n^{(u)} < 1$, for all $u > 0$.
- If $a(\hat{\eta}_t; \widehat{\boldsymbol{\theta}}_n) > 1$ for at least one $1 \leq t \leq n$, then there exists a unique $u_n > 0$ such that $S_n^{(u_n)} = 1$.

Letting

$$\hat{u}_n = \sup\{u > 0; S_n^{(u)} \leq 1\},$$

we have $\hat{u}_n = \infty$ when $a(\hat{\eta}_t; \widehat{\boldsymbol{\theta}}_n) \leq 1$ for all $1 \leq t \leq n$, and $\hat{u}_n = u_n$ in the opposite case.

Asymptotic distribution of the empirical MME

Under the previous (and additional) assumptions

If $P[a(\eta_1) > 1] > 0$, and $1 < E[a^s(\eta_1)] < \infty$ for some $s > 0$, then

$$\hat{u}_n \to u_0, \quad a.s.$$

and

$$\sqrt{n}(\hat{u}_n - u_0) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, w_{u_0}^2 := \{D_\infty^{(u_0)}\}^{-2} v_{u_0}^2\right),$$

where $D_\infty^{(u_0)} := \frac{\partial}{\partial u} S_\infty^{(u_0)}$.

Remark: The proof is based on the weak convergence of the empirical MGF, on the space $\mathcal{C}$ equipped with the uniform distance. For $[u_1, u_2] \subset (0, s/2)$

$$\sqrt{n}\left\{S_n^{(u)} - S_\infty^{(u)}\right\} \overset{\mathcal{C}[u_1,u_2]}{\Longrightarrow} \Gamma(u),$$

where $\Gamma(u)$ is a centered Gaussian process.

Test based on the empirical MME

Noting that the null assumption of finite $u\delta$-th moment can be written $\boldsymbol{H}_{0,u}: \quad u < u_0$, let the test statistic,

$$U_n^{(u)} = \frac{\sqrt{n}\,\{u - \hat{u}_n\}}{\widehat{w}_{\hat{u}_n}},$$

where $\widehat{w}_u^2 = \left\{\frac{1}{n}\sum_{t=1}^n a^{\hat{u}_n}(\hat{\eta}_t; \widehat{\boldsymbol{\theta}}_n)\log\{a(\hat{\eta}_t; \widehat{\boldsymbol{\theta}}_n)\}\right\}^{-2} \hat{v}_u^2$.

Test of $\boldsymbol{H}_{0,u}$ [resp. $\boldsymbol{H}_{0,u}^*$] at the asymptotic level $\underline{\alpha} \in (0,1)$

$$C_U^{(u)} = \{U_n^{(u)} > \Phi^{-1}(1-\underline{\alpha})\}, \qquad [\text{resp. } \{U_n^{(u)} < \Phi^{-1}(\underline{\alpha})\}],$$

Purely parametric estimator of the MME

When the density f of η_t is known, the MME can be obtained by solving

$$\int a^{u_0}(x;\boldsymbol{\theta})f(x)dx = 1,$$

with solution $u_0 = u_{0,f}(\boldsymbol{\theta})$ (unique by the convexity of the MGF).

Let $\hat{u}_{n,f} = u_{0,f}(\widehat{\boldsymbol{\theta}}_{n,ML})$ where $\widehat{\boldsymbol{\theta}}_{n,ML}$ is the MLE of $\boldsymbol{\theta}_0$.

Assume that the MLE satisfies the following expansion

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_{n,ML} - \boldsymbol{\theta}_0) = \frac{2\boldsymbol{J}^{-1}}{\iota_f\sqrt{n}} \sum_{t=1}^{n} \frac{1}{\sigma_t^2} \frac{\partial \sigma_t^2}{\partial \boldsymbol{\theta}} g_1(\eta_t) + o_P(1).$$

(see Berkes, Horváth and Kokoszka (2004))

Purely parametric estimator of the MME

Let the test statistic $V_n^{(u)} = \frac{\sqrt{n}(u - \hat{u}_{n,f})}{\hat{\sigma}_f}$ where $\hat{\sigma}_f$ is a consistent estimator of $\sigma_f = \left(\frac{4}{\iota_f} \frac{\partial u_0}{\partial \boldsymbol{\theta}'} \boldsymbol{J}^{-1} \frac{\partial u_0}{\partial \boldsymbol{\theta}} \right)^{1/2}$.

Under the previous assumptions and if $\frac{\partial u_0}{\partial \boldsymbol{\theta}} \neq \mathbf{0}$,

a test of $\boldsymbol{H}_{0,u}$ [resp. $\boldsymbol{H}^*_{0,u}$] at the asymptotic level $\underline{\alpha} \in (0,1)$ is defined by

$$C_V^{(u)} = \{V_n^{(u)} > \Phi^{-1}(1 - \underline{\alpha})\}, \qquad [\text{resp. } \{V_n^{(u)} < \Phi^{-1}(\underline{\alpha})\}].$$

Test statistics for $\boldsymbol{H}_{0,u}$: $E\{a^u(\eta_t)\} < 1$

Based on the empirical MGF:

$$T_n^{(u)} = \frac{\sqrt{n}\left\{S_n^{(u)} - 1\right\}}{\hat{v}_u}$$

Based on the empirical MME:

$$U_n^{(u)} = \frac{\sqrt{n}\left\{u - \hat{u}_n\right\}}{\widehat{w}_{\hat{u}_n}}$$

Fully parametric:

$$V_n^{(u)} = \frac{\sqrt{n}(u - \hat{u}_{n,f})}{\hat{\sigma}_f}$$

Asymptotic power under local alternatives

Around $\boldsymbol{\theta}_0 \in \overset{\circ}{\Theta}$, let a sequence of local parameters of the form

$$\boldsymbol{\theta}_n = \boldsymbol{\theta}_0 + \boldsymbol{\tau}/\sqrt{n},$$

where $\boldsymbol{\tau} \in \mathbb{R}^d$.

Let $P_{n,\boldsymbol{\tau}}$ (resp. P_0) the distribution of the observations when the parameter is $\boldsymbol{\theta}_0 + \boldsymbol{\tau}/\sqrt{n}$ (resp. $\boldsymbol{\theta}_0$).

Under appropriate assumptions on $\boldsymbol{\tau}$, the parameter $\boldsymbol{\theta}_n$ belongs to the alternative for testing $\boldsymbol{H}_{0,u_0}$.

Local asymptotic powers

LAP of the tests T, U and V

$$\lim_{n\to\infty} P_{n,\boldsymbol{\tau}}\left(C_T^{(u_0)}\right) = \lim_{n\to\infty} P_{n,\boldsymbol{\tau}}\left(C_U^{(u_0)}\right) = \Phi\left\{\mathbf{c}_{\mathbf{f},\mathbf{u_0}}(\theta_{\mathbf{0}}) - \Phi^{-1}(1-\underline{\alpha})\right\},$$

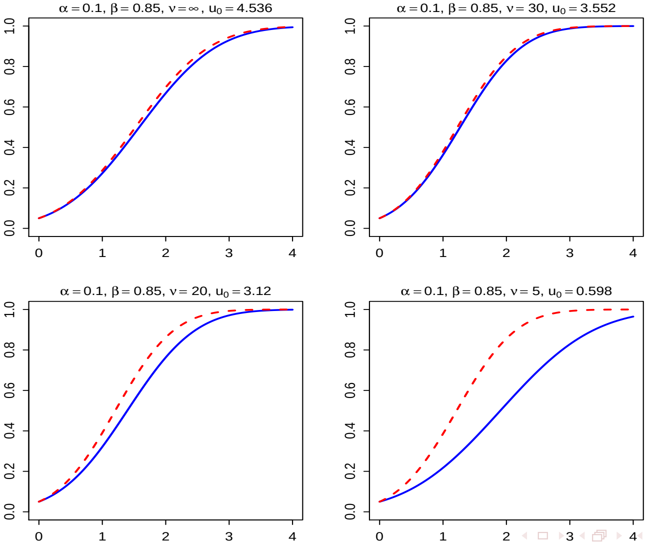
$$\lim_{n\to\infty} P_{n,\boldsymbol{\tau}}\left(C_V^{(u_0)}\right) = \Phi\left\{\mathbf{d}_{\mathbf{f},\mathbf{u_0}}(\theta_{\mathbf{0}}) - \Phi^{-1}(1-\underline{\alpha})\right\},$$

where, using $g_1(y) = 1 + y\frac{f'}{f}(y)$ and $\boldsymbol{r}_{u_0} = \frac{\partial}{\partial\boldsymbol{\theta}} S_\infty^{u_0}(\boldsymbol{\theta}_0)$,

$$\begin{aligned} c_{f,u_0}(\boldsymbol{\theta}_0) = &-\frac{\boldsymbol{\tau}'}{v_{u_0}}\left[E\left(\frac{1}{\sigma_t}\frac{\partial\sigma_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\right)E\{a^{u_0}(\eta_1)g_1(\eta_1)\}\right. \\ &\left.+E\left(\frac{1}{\sigma_t}\frac{\partial\sigma_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\boldsymbol{g}'_{u_0}\boldsymbol{\Delta}_{t-1}\right)E\{\boldsymbol{V}(\eta_1)g_1(\eta_1)\}\right], \end{aligned}$$

$$d_{f,u_0}(\boldsymbol{\theta}_0) = \frac{\boldsymbol{r}'_{u_0}\boldsymbol{\tau}}{\sqrt{\frac{4}{\iota_f}\boldsymbol{r}'_{u_0}\boldsymbol{J}^{-1}\boldsymbol{r}_{u_0}}}.$$

LAPs of the test T, U (blue line) and V (dotted red line) for a GARCH(1,1) with Student errors



Comparisons based on Bahadur slopes

To be able to distinguish the tests T and U, the Bahadur approach can be used.

slope $=$ a.s limit of $-2/n\times$ the logarithm of the p-value under $P_{\boldsymbol{\theta}}$

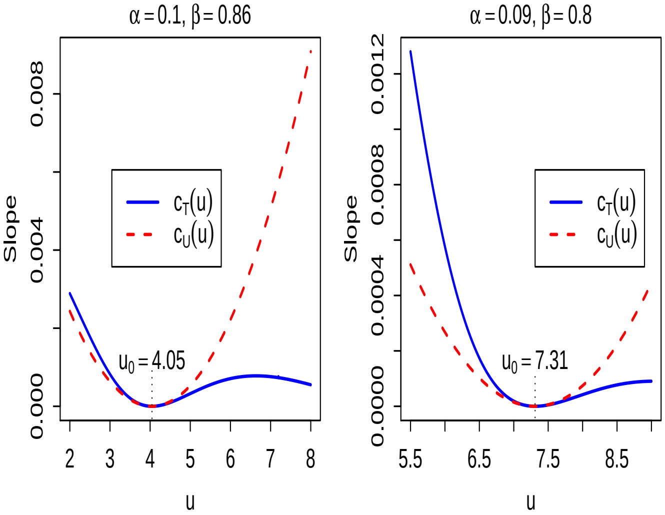
Asymptotic slopes of the tests:

$$c_T(u) = \frac{\left\{S_\infty^{(u)} - 1\right\}^2}{v_u^2} \quad \text{and} \quad c_U(u) = \frac{\{u-u_0\}^2}{{w_{u_0}}^2}.$$

In the Bahadur sense, $T_n^{(u)}$ is more efficient than $U_n^{(u)}$ iff

$$\frac{c_T(u)}{c_U(u)} = \frac{\left\{S_\infty^{(u)} - 1\right\}^2}{\{u-u_0\}^2} \frac{v_{u_0}^2}{\{E[a^{u_0}(\eta_1;\boldsymbol{\theta}_0)\log\{a(\eta_1;\boldsymbol{\theta}_0)\}]\}^2 v_u^2} > 1.$$

Bahadur slopes of the tests T and U for GARCH(1,1) models with Gaussian errors



Conclusions

- Tests based on the QML are valid whatever the distribution of the innovations. However, choosing the appropriate version of the QML can bring efficiency gains without much additional cost.
- The bootstrap versions of our tests bring significant improvements in terms of size but, as expected, do not improve powers.
- Locally optimal tests are worth considering both in terms of size and power, but may be inconclusive for moderate sample sizes.
- Augmented GARCH(1,1) can also be tested at any power.
- Tests can be extended to a parametrized error density (not shown).
- Numerical results suggest that one has to be cautious in assessing the existence, or non-existence, of moments of financial time series.

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Thank you!

Resampling scheme for $m = 1$ (2nd-order stationarity)

In the GARCH(1,1) case:

1. Compute the constrained QMLE

$$\hat{\boldsymbol{\theta}}_c' = (\hat{\omega}_c, \hat{\alpha}_c, 1 - \hat{\alpha}_c) = \arg\min_{\boldsymbol{\theta} \in \Theta^c} \sum_{t=1}^{n} \tilde{\ell}_t(\boldsymbol{\theta})$$

   and the standardized residuals $\hat{\eta}_t = \tilde{\eta}_t / s_n$, where $\tilde{\eta}_t = \epsilon_t / \tilde{\sigma}_t(\hat{\boldsymbol{\theta}}_c)$ and $s_n^2 = n^{-1} \sum_{t=1}^{n} \tilde{\eta}_t^2$. Denote by F_n^* the empirical distribution of these residuals.

2. Simulate a trajectory of length n of a GARCH model with the parameter $\hat{\boldsymbol{\theta}}_c$ and distribution F_n^* for the i.i.d. noise η_t^*, compute the unconstrained QMLE $\hat{\boldsymbol{\theta}}^* = (\hat{\omega}^*, \hat{\alpha}^*, \hat{\beta}^*)'$ and the statistic $S_n^* = \hat{\alpha}^* + \hat{\beta}^*$

3. On the observations $\epsilon_1, \dots, \epsilon_n$, compute the unconstrained QMLE $\hat{\boldsymbol{\theta}} = (\hat{\omega}, \hat{\alpha}, \hat{\beta})$ and the statistic $S_n = \hat{\alpha} + \hat{\beta}$

4. Repeat B times step 2, and denote by $S_n^{*1}, \dots, S_n^{*B}$ the bootstrap test statistic.

   Approximate the p-value of the test of $H_0 : E\epsilon_t^2 < \infty$ against $H_1 : E\epsilon_t^2 = \infty$ by $\#\{S_n^{*j} \geq S_n; j = 1, \dots, B\}/B$.

Resampling scheme for $m = 1$ using Newton-Raphson

The numerical optimization in Step 2, repeated a large number of times B, is the most time-consuming part of the algorithm.

Instead, one can mimic the distribution of the QMLE by using a Newton-Raphson type iteration. Set

$$\hat{\boldsymbol{\theta}}^* = \hat{\boldsymbol{\theta}}_c + \boldsymbol{J}_n^{-1} \frac{1}{n} \sum_{t=1}^{n} \left(\eta_t^{*\,2} - 1\right) \tilde{\phi}_t(\hat{\boldsymbol{\theta}}_c),$$

where

$$\tilde{\phi}_t(\boldsymbol{\theta}) = \frac{1}{\tilde{\sigma}_t(\boldsymbol{\theta})} \frac{\partial \tilde{\sigma}_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}, \qquad \boldsymbol{J}_n = \frac{1}{n} \sum_{t=1}^{n} \tilde{\phi}_t \tilde{\phi}_t'(\hat{\boldsymbol{\theta}}_c)$$

and $\eta_1^*, \ldots, \eta_n^*$ are independent and F_n^*-distributed.

Validity of the resampling scheme for testing $H_0 : E\epsilon_t^2 < \infty$ (or $H_0^* : E\epsilon_t^2 = \infty$)

Let $\boldsymbol{\theta}_0$ such that $\boldsymbol{c}'\boldsymbol{\theta}_0 = 1$ with $\boldsymbol{c}' = (0, 1, \ldots, 1)$.

Asymptotic validity of the bootstrap procedure for the GARCH(p, q)

Assume **A1-A4** + a bounded density for η_t.
Let $\hat{\boldsymbol{\theta}}^*$ obtained in Step 2 (or by a NR iteration).
For almost all realization (ϵ_t), as $n \to \infty$ we have, given (ϵ_t),

$$\sqrt{n}\,(S_n^* - 1) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2), \qquad \sigma^2 = (\mu_4 - 1)\boldsymbol{c}'\boldsymbol{J}^{-1}\boldsymbol{c}.$$

$\Rightarrow$ the law of S_n^* given (ϵ_t) well mimics the (unconditional) law of S_n at the boundary of H_0, at least for large n.

In finite samples, the bootstrap distribution of S_n^* is expected to better approach the law of S_n than its asymptotic distribution.

Bootstrap procedure for testing the existence of $E\epsilon_t^{2m}$ when $m > 1$

5. Estimate a GARCH(1,1) model and compute $\hat{\mu}_{2i} = n^{-1}\sum_{t=1}^{n}\hat{\eta}_t^{2i}$ on the recentred and rescaled residuals.
6. Estimate a GARCH(1,1) model of parameter $\boldsymbol{\theta}_c = (\omega_c, \alpha_c, \beta_c)$ under the constraint $H_0 : \sum_{i=0}^{m}\binom{m}{i}\alpha_c^i\beta_c^{m-i}\hat{\mu}_{2i} = 1$.
7. Simulate a trajectory of length n of a GARCH model with the parameter $\hat{\boldsymbol{\theta}}_c$ of the previous step, and the empirical distribution of the unconstrained residuals for the i.i.d. noise. Compute the unconstrained QMLE $\hat{\boldsymbol{\theta}}^* = (\hat{\omega}^*, \hat{\alpha}^*, \hat{\beta}^*)'$ and the statistic $S_n^* = \sum_{i=0}^{m}\binom{m}{i}\hat{\alpha}^{*\,i}\hat{\beta}^{*\,m-i}\hat{\mu}_{2i}^*$ where $\hat{\mu}_{2i}^*$ is computed on the residuals based on $\hat{\boldsymbol{\theta}}^*$.
8. Compute $S_n = \sum_{i=0}^{m}\binom{m}{i}\hat{\alpha}^i\hat{\beta}^{m-i}\hat{\mu}_{2i}$.
9. As Step 4.

Remark: Heinemann (2019) establishes the validity of a *fixed-design* residual bootstrap (as in Cavaliere, Pedersen and Rahbek (2018)) for testing the existence of moments for GARCH(p,q) processes.

Return