Zero-sum squares in bounded discrepancy $\{-1, 1\}$–matrices

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Joint work with Alma R. Arévalo and Edgardo Roldán-Pensado

Additive Combinatorics in Marseille, HYBRID CONFERENCE 2020

Outline of the talk

1. The Erickson matrices problem (Ramsey Theory)
2. Zero-sum Ramsey Theory
3. Zero-sum squares in bounded discrepancy $\{-1, 1\}$–matrices

Erickson matrices

- An *Erickson matrix* is a square binary matrix that contains no *squares* with constant entries.
- A *square* S in $M = (a_{i,j})$ is a 2×2 sub-matrix of M of the form

$$S = \begin{pmatrix} a_{i,j} & a_{i,j+s} \\ a_{i+s,j} & a_{i+s,j+s} \end{pmatrix}$$

for some positive integer s.

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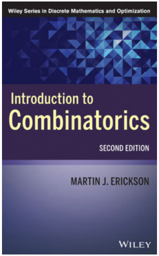
How large can an Erickson matrix be? (1996)



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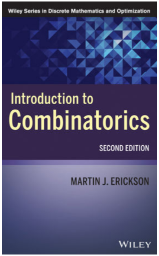

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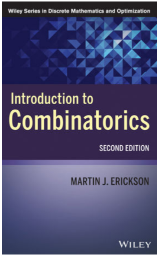

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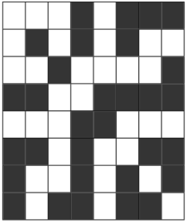

Rephrasing the problem

- A square binary matrix is a 2-coloring of the grid $[n] \times [n]$.
- A constant square is a monochromatic square.

Problem (Erickson, 1996)

Determine the minimum positive integer $R_2(S)$ such that every 2-coloring of $[n] \times [n]$, with $n \geq R_2(S)$, contains a monochromatic square.

Solution of the problem

- $13 \leq R_2(S) \leq \min\{w(2;8), 5 \cdot 2^{2^{40}})$ (Axenovich and Manske, 2008).

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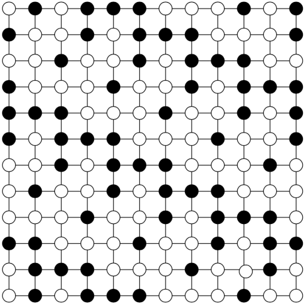

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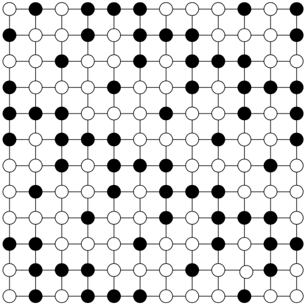

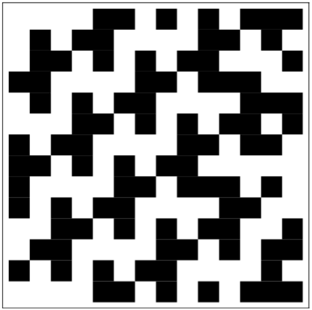

- $R_2(S) = 15$ (Bacher and Eliahou, 2010).

$$f : \mathcal{U} \to \{c_1, c_2, \ldots, c_k\}$$

$$\mathcal{U} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_k \text{ where } \mathcal{C}_i = f^{-1}(i)$$

- Ramsey Theory: study monochromatic subsets.
- Anti-Ramsey Theory: study rainbow subsets.
- Zero-Sum Ramsey Theory: study zero-sum subsets.

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When bounded discrepancy (that is, $|\sum_{x \in \mathcal{U}} f(x)|$) implies the existence of a zero-sum substructure?

For example

$$+ + - + + - - - + - + - - + + - + - + - - + + - + - + -$$

$$f : [n] \to \{-1, 1\}$$

Q1: Is it true that every (sufficiently large) bounded discrepancy $\{-1, 1\}$-sequence contains a zero-sum arithmetic progression?

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Yes!

Zero-sum k-blocks

Theorem (Caro, Hansberg, M, 2019)

Let t, k and q be integers such that $q \geq 0$, $0 \leq t < k$, and $t \equiv k \pmod 2$, and take $s \in [0, t+1]$ as the unique integer satisfying $s \equiv q + \frac{k-t-2}{2} \pmod{(t+2)}$. Then, for any integer

$$n \geq \frac{1}{2(t+2)}k^2 + \frac{q-s}{t+2}k - \frac{t}{2} + s$$

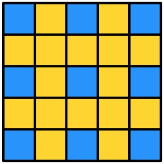
and any function $f : [n] \to \{-1, 1\}$ with $|f([n])| \leq q$, there is a k-block $B \subseteq [n]$ with $|f(B)| \leq t$.

Zero-sum squares in $\{-1,1\}$–matrices

$$f : [n] \times [n] \to \{-1,1\}$$

Q2: Is it true that every (sufficiently large) bounded discrepancy $\{-1,1\}$-matrix contains a zero-sum square?

$$\begin{pmatrix} + & - & + & - & + \\ - & - & - & - & - \\ + & - & + & - & + \\ - & - & - & - & - \\ + & - & + & - & + \end{pmatrix}$$



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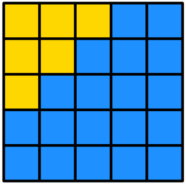

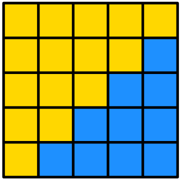

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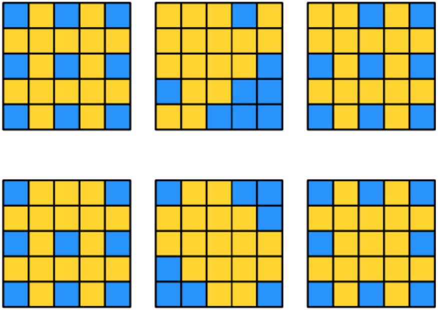
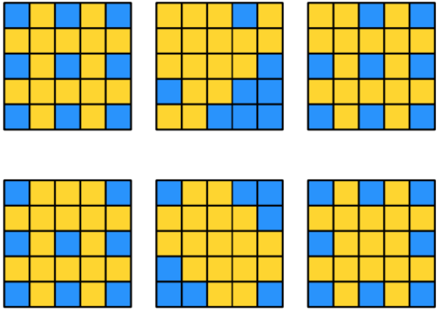


With computing assistance we observe that

for $5 \leq n \leq 11$, every $n \times n$ non-triangular $\{-1, 1\}$-matrix M with disc$(M) \leq n$ contains a zero-sum square.



Theorem (Arévalo, Roldán-Pensado, M, 202+)

Let $n \geq 5$. Every $n \times n$ non-diagonal $\{-1, 1\}$-matrix M with $\text{disc}(M) \leq n$ contains a zero-sum square.

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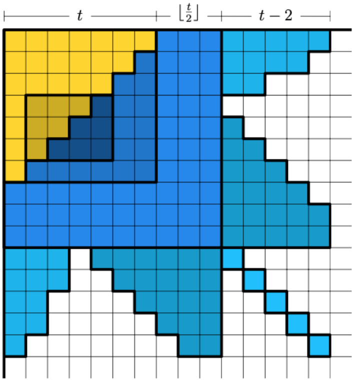

Theorem (Arévalo, Roldán-Pensado, M, 202+)

Let $n \geq 5$ and $m \in \{n, n+1\}$. Every $n \times m$ non-diagonal $\{-1, 1\}$-matrix M with $\operatorname{disc}(M) \leq n$ contains a zero-sum square.

Open problems

- Other ranges, other shapes, larger dimension, etc.

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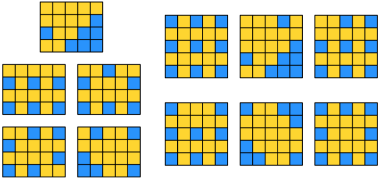
For $5 \leq n \leq 11$, these are all examples (up to symmetries) of $n \times n$ and $n \times (n+1)$ non-triangular zero-sum-square-free $\{-1, 1\}$-matrices M with $\mathrm{disc}(M) \leq 2n$.



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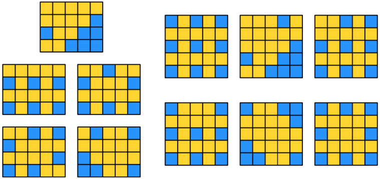

Conjecture (Arévalo, Roldán-Pensado, M, 202+)

For every $C > 0$ there is a integer N such that whenever $n \geq N$ the following holds: Every $n \times n$ non-diagonal $\{-1, 1\}$-matrix M with disc(M)$\leq Cn$ contains a zero-sum square.

Thank you for your attention!!