Herglotz–Nevanlinna functions: Realizations and Generalizations

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Recall:

Definition: A function $h : \mathbb{C}^+ \to \mathbb{C}$ is called Herglotz–Nevanlinna function if it is analytic and $\mathrm{Im} h(z) \geq 0$ (for $\mathrm{Im} z > 0$) .

Theorem [Nevanlinna, Herglotz, Pick, Cauer....]
A function h: $\mathbb{C}^+ \to \mathbb{C}$ is a Herglotz-Nevanlinna function $\Longleftrightarrow$

$$h(z) = a + bz + \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \mathrm{d}\mu(t)$$

where $a \in \mathbb{R}$, $b \geq 0$ and μ is a positive Borel measure with $\int\limits_{-\infty}^{\infty} \frac{1}{1+t^2} \mathrm{d}\mu(t) < \infty$.
Moreover, a, b and μ are unique in this representation.

Examples

Symmetric extension: $\widetilde{h}(z) := \begin{cases} h(z) & z \in \mathbb{C}^+ \\ \overline{h(\overline{z})} & z \in \mathbb{C}^- \end{cases}$ Note $\widetilde{h}(\overline{z}) = \overline{\widetilde{h}(z)}$.

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$\mathcal{H}$ **Hilbert space**, $v \in \mathcal{H}$

A **self-adjoint(multivalued) operator**

$z_0 \in \mathbb{C}^+$.

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A **self-adjoint(multivalued) operator**

$z_0 \in \mathbb{C}^+$,

(A) $\Rightarrow$ (D): Herglotz-.... Theorem

(D) $\Rightarrow$ (E): e.g. space L^2_μ, A "multiplication by t"

(E) $\Rightarrow$ (A): straight forward

Remark: alternative operators: second order differential operator, canonical systems,....

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Sum rule - one case

Definition: h has an asymptotic expansion of order $2N_\infty - 1$ at $z = \infty$ if there exist *real* numbers $b_1, b_0, b_{-1}, \ldots, b_{-(2N_\infty - 1)}$ such that

$$h(z) = b_1 z + b_0 + \frac{b_{-1}}{z} + \ldots + \frac{b_{-(2N_\infty-1)}}{z^{2N_\infty-1}} + o\left(\frac{1}{z^{2N_\infty-1}}\right) \qquad \text{as } z \hat{\to} \infty.$$

A sum rule: Let h be an Herglotz-Nevanlinna function. Then

$$p.v. \int_{\mathbb{R}} x^n \, \mathrm{Im} h(x + i0^+) \, dx = -b_{-n-1}.$$

for $0 < n \leq 2N_\infty - 2$.

More precisely:
For some integer $N_\infty \geq 1$ the limit

$$\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \int_{\varepsilon < |x| < \frac{1}{\varepsilon}} x^{2(N_\infty - 1)} \, \mathrm{Im} h(x + iy) \, dx$$

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Definition[Krein,Langer; 1970ies] A function $q : \mathcal{D} \subset \mathbb{C} \to \mathbb{C}$, which

- q is meromorphic in $\mathbb{C} \setminus \mathbb{R}$
- the kernel N_q has κ negative squares

is called Generalized Nevanlinna function, $q \in \mathcal{N}_\kappa$.

Examples

Towards generalizations

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[Krein,Langer77]

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[Dijksma,Langer,AL,Shondin00; Derkach,Hassi,deSnoo,99]

Sum rules do in general not work!

Question: interesting subclasses???

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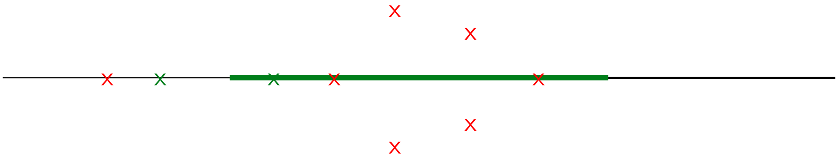

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[Ivanenko,Nedic,Gustafsson,Jonsson,AL,Nordebo20, AL,Nedic22(?)]

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Question: Characterize the operators??

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Theorem[Delsarte,Genin,Kamp86] These functions are exactly those of the form

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A self-adjoint with additional property such as:

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integral representation, operator representation, transfer function approach work!

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Survey book-chapter

to appear in AWM Volume "Research in the Mathematics of Materials Science"

- AL and Miao-Jung Yvonne Ou, *On applications of Herglotz-Nevanlinna functions in material sciences, I: classical theory and applications of sum rules*

- Miao-Jung Yvonne Ou and AL, *On applications of Herglotz-Nevanlinna functions in material sciences, II: extended applications and generalized theory*

THANK YOU!

Survey book-chapter

to appear in AWM Volume "Research in the Mathematics of Materials Science"

- AL and Miao-Jung Yvonne Ou, *On applications of Herglotz-Nevanlinna functions in material sciences, I: classical theory and applications of sum rules*

- Miao-Jung Yvonne Ou and AL, *On applications of Herglotz-Nevanlinna functions in material sciences, II: extended applications and generalized theory*

THANK YOU!