Curve neighbourhoods for odd symplectic Grassmannians

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(joint with Ryan Shifler, arXiv:2011.05221)

## WHAT ARE CURVE NEIGHBOURHOODS?

- introduced by Buch-Chaput-Mihalcea-Perrin (2013) to prove finiteness of quantum K-theory for cominuscule homogeneous spaces;
- $\left.\begin{array}{l} V \subseteq X \text{ subvariety} \\ d \in \mathbb{Z}_{>0} \end{array}\right\} \rightsquigarrow \Gamma_d(V) = \overline{\bigcup \left\{ \begin{array}{l} \text{degree } d \text{ rational curves} \\ \text{through } V \end{array} \right\}}$

## WHY ARE THEY USEFUL?

- applications to quantum K-theory (see BCMP 2013)
- applications to quantum cohomology (see e.g. Buch-Mihalcea 2015)

# WHAT ARE ODD SYMPLECTIC GRASSMANNIANS?

- a family of quasi-homogeneous spaces with homogeneous-like behaviour (e.g., Schubert-type subvarieties);
- studied in Mihai 2007, Pasquier 2009, P. 2013, Gonzales-P.-Perrin-Samokhin 2019, Mihalcea-Shifler 2019, Li-Mihalcea-Shifler 2019

Definition: $E = \mathbb{C}^{2n+1}$; $\left.\begin{array}{l}\omega \in \Lambda^2 E^* \\ \dim \operatorname{Ker} \omega = 1\end{array}\right\} \rightsquigarrow \omega$ odd symplectic form; $1 \leq k \leq n$

$$X := \mathrm{IG}(k, E) = \{ V \subseteq E \mid \dim V = k;\ \omega_{|V} \equiv 0 \}$$

$\rightsquigarrow$ odd symplectic Grassmannian

Remark: $\exists\ F^{2n} \subseteq E^{2n+1} \subseteq \tilde{E}^{2n+2}$ such that: $\begin{cases} \omega_{|F} \text{ symplectic} \\ \tilde{\omega} \text{ symplectic on } \tilde{E} \text{ and } \omega = \tilde{\omega}_{|E} \end{cases}$

Then: $\mathrm{IG}(k-1, F) \subseteq \mathrm{IG}(k, E) \subseteq \mathrm{IG}(k, \tilde{E}) = X^{ev}$

# TWO INTERESTING GROUP ACTIONS ON $X = IG(k, 2n+1)$

## ① The action of $\mathrm{Aut}(X) = Sp_{2n+1}$, the odd symplectic group.

- $Sp_{2n+1} = \{ g \in GL_{2n+1} \mid {}^t g \omega g = \omega \}$ (cf Proctor 1988)
  ⚠ non reductive
- $IG$ has two $Sp_{2n+1}$-orbits when $1 < k < n$:

$$\begin{cases} Z = \{ V \in X \mid V \supseteq \mathrm{Ker}\,\omega \} \simeq IG(k-1, 2n) \rightsquigarrow \text{closed orbit} \\ X^\circ = X \setminus Z \end{cases}$$

## ② The action of $Sp_{2n}$, the symplectic group.

- $Sp_{2n} = Sp(F) \rightsquigarrow$ horospherical action (cf Pasquier 2009)
- $IG$ has three $Sp_{2n}$-orbits when $1 < k < n$:

$$\begin{cases} Z,\ Y = IG(k, F) \rightsquigarrow \text{closed orbits} \\ U = X \setminus (Y \cup Z) \cong (\mathbb{C}^*)^k\text{-bundle over } IG(k, 2n) \end{cases}$$

Theorem. $X(\lambda) \subseteq X$ Schubert variety ($\lambda \in$ indexing set), $d \in \mathbb{Z}_{\geq 0}$.

$$\Gamma_d(X(\lambda)) = \begin{cases} X(\lambda^{0^\circ(d)}) & \text{if } X(\lambda) \cap X^\circ \neq \emptyset; \\ X(\lambda^{0_Y(d)}) \cup X(\lambda^{0_Z(d)}) & \text{if } X(\lambda) \subseteq Z;\ \lambda \in \mathrm{Comp}(d); \\ X(\lambda^{0_Y(d)}) & \text{if } X(\lambda) \subseteq Z;\ \lambda \notin \mathrm{Comp}(d). \end{cases}$$

Remarks

- 2 possible indexing sets, BKT partitions & BC partitions
- $X(\lambda^{0_Z(d)}) \subseteq Z$ ; $X(\lambda^{0^\circ(d)})$, $X(\lambda^{0_Y(d)}) \cap X^\circ \neq \emptyset$
- ⚠ $\Gamma_d(X(\lambda))$ is not always irreducible (unlike for homogeneous spaces)

**Proposition** [Mihai 2007] $IG(k,2n+1) \hookrightarrow IG(k,2n+2)$ identifies $IG(k,2n+1)$ with a (smooth) Schubert variety of $IG(k,2n+2)$

$\rightsquigarrow$ $IG(k,2n+1)$ has a Schubert decomposition + corresponding indexation by partitions.

**BKT partitions** [Buch-Kresch-Tamvakis 2009]

- $\beta \subseteq {}^{k}\overset{2n+2-k}{\boxed{\qquad\quad}}$ is $(n+1-k)$-strict if $\beta_i > n+1-k \Rightarrow \beta_i > \beta_{i+1}$

$\rightsquigarrow \beta \in BKT(k,2n+2)$

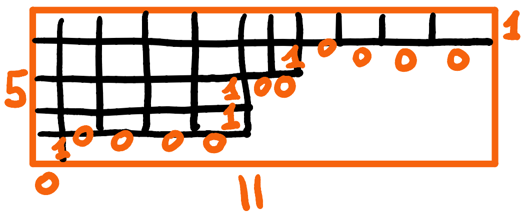
- $BKT(k,2n+1) = \{ \beta - {}^{k}\boxed{\vdots} \mid \beta \in BKT(k,2n+1) \}$

ADVANTAGE: $\beta \in BKT(k,2n+1) \Rightarrow \operatorname{codim} X(\beta) = |\beta|$

DRAWBACK: $X(\alpha) \subseteq X(\beta) \;\cancel{\Rightarrow}\; \alpha \geq \beta$

## BC-partitions [Shifler-Withrow 2020]

$\mu \subseteq k \underbrace{\boxed{\phantom{xxxxxxxx}}}^{2n+2-k} \rightsquigarrow$ 01-word $D(\mu)$

e.g. $\mu =$  $\rightsquigarrow D(\mu) = 1000010011000010$

**Definition:**

- $\mu \in BC(k, 2n+2)$ means: if $\exists\, 1 \leq i \leq n+1$ s.t. $D(\mu)(i) = D(\mu)(2n+3-i)$, then $D(\mu)(i) = 0$.
- $BC(k, 2n+1) = \left\{ \mu \subseteq k \underbrace{\boxed{\phantom{xxxxxxxx}}}_{2n+1-k} \;\middle|\; \mu + k\boxed{\vdots} \in BC(k, 2n+1) \right\}$

ADVANTAGE: $\lambda, \mu \in BC(k, 2n+1)$ & $X(\lambda) \subseteq X(\mu) \Rightarrow \lambda \supseteq \mu$

DRAWBACK: $\operatorname{codim} X(\lambda) \neq |\lambda|$ in general.

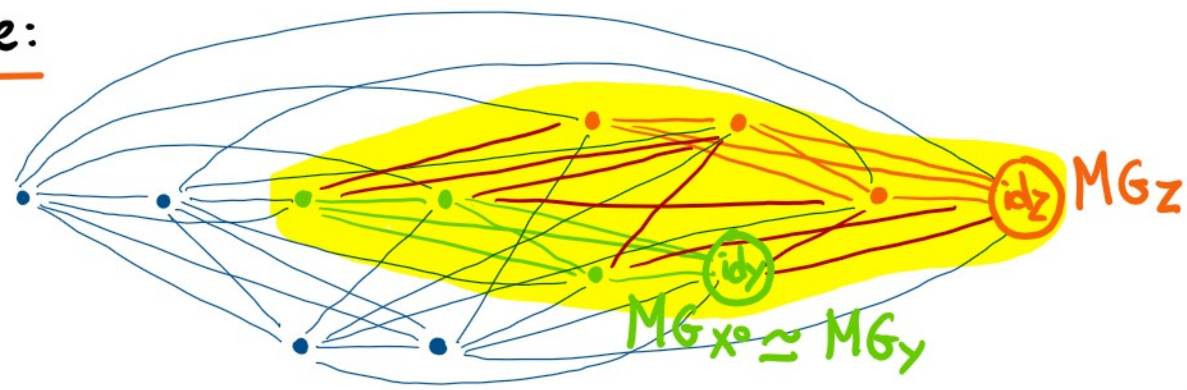
# THE MOMENT GRAPH $MG_X$

⇝ encodes data about curve neighbourhoods

⇝ vertices = torus-fixed points
edges = torus-stable curves (deg = 1,2 for symplectic Grassmannians)

Proposition: ① $MG_X \subseteq MG_{X^{ev}}$ as a full subgraph
② $MG_{X^o}$ (full subgraph of $MG_X$ induced by T-fixed points in $X^o$) is isomorphic to $MG_Y$ ($Y = IG(k, 2n)$)

Example:




$X^{ev} = IG(2,6)$
$X = IG(2,5)$
$Y = IG(2,4)$
$Z = IG(1,4)$

$= MG_X$

# PROOF OF MAIN RESULT

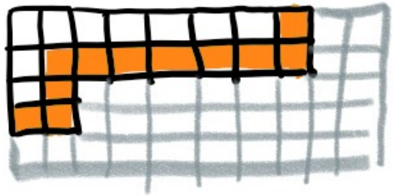
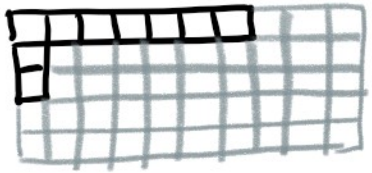
① Curve neighbourhoods in $X^\circ$:
- $X(w) \cap X^\circ \neq \emptyset \Rightarrow \Gamma_d(X(w))$ irred [BM 2015]
- Compute $\Gamma_d(X(id_Y))$
- Use $Sp_{2n}$-action for $\Gamma_d(X(w))$

② Curve neighbourhoods in $Z$:
- ⚠ not always irreducible
- Compute $\Gamma_d(X(id_Z))$ $(X(id_Z) = \text{point})$
  ⇝ two irreducible components
- Use $Sp_{2n}$-action for general case

Remark:
- $\Gamma_d(X(w)) = ev_2(ev_1^{-1}(X(w))) \subseteq X$, $ev_i : \underbrace{\mathcal{M}_{0,2}(X,d)}_{\text{moduli space of stable maps}} \longrightarrow X$
- group action allows to focus on $T$-fixed points, then use moment graph results.

# COMBINATORIAL DESCRIPTION USING BC-PARTITIONS

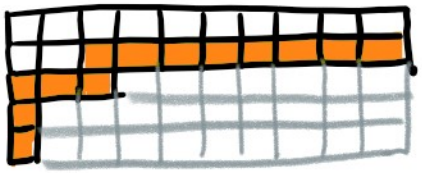
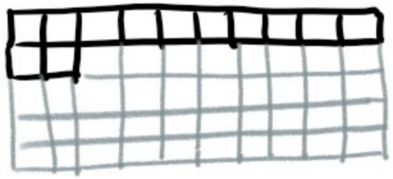
① $X(\lambda) \cap X^{\circ} \neq \emptyset$: $\Gamma_d(X(\lambda)) = X(\lambda^{\circ\circ(d)})$, where $\begin{cases} \lambda^{\circ\circ(1)} = \overbrace{(\lambda_2 - 1, \dots, \lambda_{e(\lambda)} - 1)}^{\lambda\text{-hook}} \\ \lambda^{\circ\circ(d)} = (\lambda^{\circ\circ(d-1)})^{\circ\circ(1)} \end{cases}$

e.g. $\lambda =$  $\rightsquigarrow \lambda^{\circ\circ(1)} =$ 

② $X(\lambda) \subseteq Z$: $\Gamma_d(X(\lambda)) = X(\lambda^{\circ y(d)}) \cup X(\lambda^{\circ z(d)})$

where $\lambda^{\circ z(1)} = (\lambda_1, \lambda_3 - 1, \dots, \lambda_{e(\lambda)} - 1)$ (remove hook from rows 2 → end)

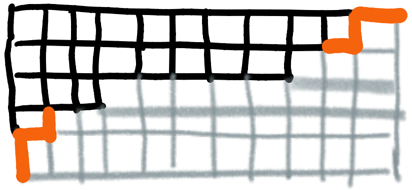
$\lambda^{\circ z(d)} = (\lambda^{\circ z(d-1)})^{\circ z(1)}$

e.g. $\lambda =$  $\rightsquigarrow \lambda^{\circ z(1)} =$ 

**Definition:** $\mu \subseteq$ $k$  $N-k$ is $m$-wingtip symmetric if

$$m = \max\{ r \in \mathbb{Z}_{\geq 0} \mid D(\mu)(i) = D(\mu^t)(i) \ \forall\, 1 \leq i \leq r \}$$

$\lambda \in BC(k, 2n+1)$ is $m$-wingtip symmetric if $\lambda +$ $k$  $\in BC(k, 2n+1)$ is.

e.g. $\mu =$  $\in BC(5,16)$ is 3-wingtip symmetric.

$(D(\mu) = 0100100000100101)$

$\lambda$ $m$-wingtip symmetric:

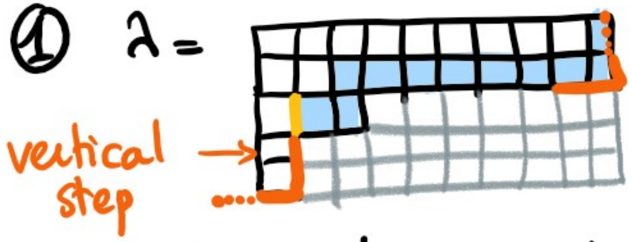
- if $D(\lambda)(2n+3-m) = 1$ is the $i^{th}$ one in $D(\lambda)$:
  $$\lambda^{or(1)} = (\lambda_2 - 1, \dots, \lambda_{i-1} - 1, \lambda_i, \lambda_i, \lambda_{i+1}, \dots, \lambda_{\ell(\lambda)})$$
- if $D(\lambda)(2n+3-m) = 0$ is a horizontal step at the bottom of the $i^{th}$ row and the $j^{th}$ column:
  $$\lambda^{or(1)} = (\lambda_2 - 1, \dots, \lambda_i - 1, j, \lambda_{i+1}, \dots, \lambda_{\ell(\lambda)})$$

$$\lambda^{or(d)} = (\lambda^{or(1)})^{\circ(d-1)}$$

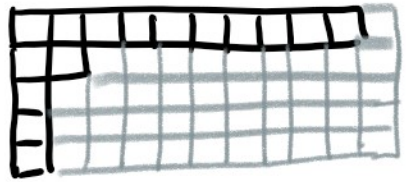
$\rightsquigarrow$ remove partial hook from $\lambda$

# COMBINATORIAL DESCRIPTION USING BC-PARTITIONS (CONT'D)

e.g. ① $\lambda =$



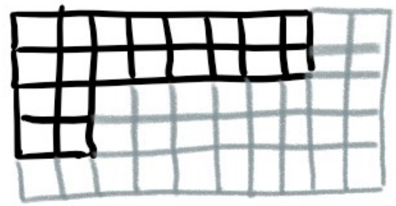

$\rightsquigarrow \lambda^{oy(1)} =$



4-wingtip symmetric

② $\lambda =$


$\rightsquigarrow \lambda^{oz(1)} =$



4-wingtip symmetric

Remark: Sometimes $\lambda^{oz(1)} \supseteq \lambda^{oy(1)} \Rightarrow X(\lambda^{oz(1)}) \leq X(\lambda^{oy(1)})$

$\rightsquigarrow$ we also have a combinatorial description.

**Theorem.** $X(\lambda) \subseteq X$ Schubert variety ($\lambda \in$ indexing set), $d \in \mathbb{Z}_{\geq 0}$.

$$\Gamma_d(X(\lambda)) = \begin{cases} X(\lambda^{0^{o}(d)}) & \text{if } X(\lambda) \cap X^{o} \neq \emptyset; \\ X(\lambda^{0_{y}(d)}) \cup X(\lambda^{0_{z}(d)}) & \text{if } X(\lambda) \subseteq Z;\ \lambda \in \mathrm{Comp}(d); \\ X(\lambda^{0_{y}(d)}) & \text{if } X(\lambda) \subseteq Z;\ \lambda \notin \mathrm{Comp}(d). \end{cases}$$

# THANK YOU !