Hamiltonian reduction for affine Grassmannian slices

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Affine Grassmannian slices

We study affine Grassmannian slices for a semisimple group G.

- $Gr = G[t, t^{-1}]/G[t]$, the affine Grassmannian.
- For any coweight μ, a point $t^\mu \in Gr$
- For λ dominant,

$$Gr^\lambda = G^\vee[[t]]t^\lambda, \quad \overline{Gr^\lambda} = \cup_{\mu \leq \lambda} Gr^\mu$$

- For μ dominant,

$$\mathcal{W}_\mu = G_1^\vee[t^{-1}]t^\mu, \quad \overline{\mathcal{W}}_\mu^\lambda = \overline{Gr^\lambda} \cap \mathcal{W}_\mu$$
$$G_1[t^{-1}] = ker(G[t, t^{-1}] \to G)$$

- For any coweight μ, $S^\mu = N_-[t, t^{-1}]t^\mu$

Example

If $G = SL_n, \lambda = n\omega_1, \mu = 0$, then $\overline{\mathcal{W}}_\mu^\lambda = \mathcal{N}_{\mathfrak{sl}_n}$

Properties of these slices

These results are due to Braverman-Finkelberg-Nakajima and K-Webster-Weekes-Yacobi.

Theorem

1. $\overline{\mathcal{W}}^{\lambda}_{\mu}$ *is an affine Poisson variety with symplectic singularities, with symplectic leaves* $\mathcal{W}^{\nu}_{\mu}$ *for* $\mu \leq \nu \leq \lambda$.
2. T *acts on* $\overline{\mathcal{W}}^{\lambda}_{\mu}$ *with fixed point* t^{μ} *and attracting locus* $\overline{Gr^{\lambda}} \cap S^{\mu}$.
3. *There is an integrable system* $\Psi : \overline{\mathcal{W}}^{\lambda}_{\mu} \to \mathbb{A}^{\rho(\lambda-\mu)}$
4. $\overline{\mathcal{W}}^{\lambda}_{\mu}$ *is the Coulomb branch of a quiver gauge theory.*
5. *The quantization* $\overline{\mathcal{W}}^{\lambda}_{\mu}$ *is a truncated shifted Yangian.*
6. $H_{top}(\overline{Gr^{\lambda}} \cap S^{\mu}) = V(\lambda)_{\mu}, \quad H_{top}(\Psi^{-1}(0)) = (\mathbb{C}[N] \otimes V(\lambda))_{\mu}$

Non dominant μ

If μ is not dominant, then $\overline{\mathcal{W}}^\lambda_\mu$ requires a more complicated definition, due to Bullimore-Dimofte-Gaiotto.

$$\mathcal{W}_\mu = U_1[t^{-1}]T_1[t^{-1}]t^\mu U_{-,1}[t^{-1}] \subset G_1[t,t^{-1}] \quad (1)$$

$$\overline{\mathcal{W}}^\lambda_\mu = \overline{G[t]t^\lambda G[t]} \cap \mathcal{W}_\mu \quad (2)$$

We can even take $\lambda = 0$ and get

$$\overline{\mathcal{W}}^0_{-\nu} = \{ \text{ based maps } \mathbb{P}^1 \to G/B \text{ of degree } \nu \ \}$$

Simplest case

$$\overline{\mathcal{W}}^0_{-\alpha_i} = T^*\mathbb{C}^\times \quad Y^0_{-\alpha_i} = D(\mathbb{C}^\times)$$

Properties of generalized slices

Theorem [BFN, KWWY, Muthiah, Krylov, Zhou]

1. $\overline{\mathcal{W}}^\lambda_\mu$ is an affine Poisson variety with symplectic singularities, the symplectic leaves are $\mathcal{W}^\nu_\mu$ for $\mu \leq \nu \leq \lambda$.
2. T acts on $\overline{\mathcal{W}}^\lambda_\mu$ with fixed point t^μ and attracting locus $\overline{Gr^\lambda} \cap S^\mu$. (If $V(\lambda)_\mu \neq 0$.)
3. There is an integrable system $\Psi : \overline{\mathcal{W}}^\lambda_\mu \to \mathbb{A}^{\rho(\lambda-\mu)}$
4. $\overline{\mathcal{W}}^\lambda_\mu$ is the Coulomb branch of a quiver gauge theory.
5. The quantization $\overline{\mathcal{W}}^\lambda_\mu$ is a truncated shifted Yangian. (No longer a subquotient of the Yangian)
6. $H_{top}(\overline{Gr^\lambda} \cap S^\mu) = V(\lambda)_\mu, \quad H_{top}(\Psi^{-1}(0)) = (\mathbb{C}[N] \otimes V(\lambda))_\mu$

Gelfand-Tsetlin and category O modules

The integrable system $\Psi : \overline{\mathcal{W}}^\lambda_\mu \to \mathbb{A}^{\rho(\lambda-\mu)}$ quantizes to a polynomial subalgebra of Y^λ_μ, which we call the *GT* subalgebra.

Example

$G = SL_n$, $Y_0^{n\omega_1} = U\mathfrak{sl}_n/Z_+$ which contains the Gelfand-Tsetlin subalgebra generated by all $Z(U\mathfrak{sl}_k)$, for $k = 2, \dots, n-1$.

We study modules for Y^λ_μ on which the *GT*-subalgebra acts locally finitely and we study category $\mathcal{O}$ for Y^λ_μ. To describe these modules we use the KLRW algebra T^λ_μ and its quotient ${}_{-}T^\lambda_\mu$.

Theorem [K-Tingley-Webster-Weekes-Yacobi]

There are equivalences

$$Y^\lambda_\mu\text{-GT mod} \cong T^\lambda_\mu\text{-mod}$$

$$Y^\lambda_\mu\text{-}\mathcal{O}\text{mod} \cong {}_{-}T^\lambda_\mu\text{-mod}$$

Corollary

We can compute the number of simple Gelfand-Tsetlin modules for $U\mathfrak{sl}_n$, solving an open problem studied by Futorny, ...

Using *transport de structure*, we obtain get a categorical action on categories of modules for truncated shifted Yagnians.

Corollary

For each simple α_i, we have a functor

$$E_i : Y_\mu^\lambda\text{-}GTmod \to Y_{\mu+\alpha_i}^\lambda\text{-}GTmod$$

categorifying the representation $\mathbb{C}[N] \otimes V(\lambda)$.

Question

How can we describe these functors without using this equivalence?

Multiplication and Comultiplication

To relate Y_μ^λ and $Y_{\mu+\alpha_i}^\lambda$, we will use multiplication maps

$$\mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} \to \mathcal{W}_{\mu_1+\mu_2}$$

and comultiplication maps

$$Y_{\mu_1+\mu_2} \to Y_{\mu_1} \otimes Y_{\mu_2}$$

introduced by Finkelberg-K-Pham-Rybnikov-Weekes.

We also will use a $\mathbb{G}_a$ action on $\mathcal{W}_\mu$ defined by $a \cdot g = x_i(a)g$. This action is Hamiltonian with moment map $\Phi_i : \mathcal{W}_\mu \to \mathbb{C}$.

Hamiltonian reduction for slices

Theorem [K-Pham-Weekes]

1. The multiplication gives an isomorphism

$$\overline{\mathcal{W}}^{\lambda}_{\mu+\alpha_i} \times \overline{\mathcal{W}}^{0}_{-\alpha_i} \xrightarrow{\sim} \Phi_i^{-1}(\mathbb{C}^\times) \subset \overline{\mathcal{W}}^{\lambda}_{\mu}$$

2. There is an isomorphism

$$\overline{\mathcal{W}}^{\lambda}_{\mu} /\!\!/_1 \mathbb{G}_a \cong \overline{\mathcal{W}}^{\lambda}_{\mu+\alpha_i}$$

3. The comultiplication gives an isomorphism

$$Y^{\lambda}_{\mu+\alpha_i} \otimes Y^{0}_{-\alpha_i} \cong Y^{\lambda}_{\mu}[\Phi_i^{-1}]$$

Unfortunately, this theorem does not seem to lead to our desired functor, since this isomorphism does not preserve the integrable system.

Higgs and Coulomb branches

G a complex reductive group, V a representation of G.
Physicists define a gauge theory from G, V and two spaces.

- Higgs branch
$$T^*V /\!\!/ G = \mu^{-1}(0)/G$$
- Coulomb branch, defined by Braverman-Finkelberg-Nakajima
$$M_C(G,V) = \operatorname{Spec} H_*\left(Maps\left(D \cup_{D^\times} D,\ [V/G]\right)\right)$$

  We also get a quantization $\mathcal{A}(G,V)$ of the Coulomb branch by taking $\mathbb{C}^\times$-equivariant homology.

Fix a semisimple group G_Q and λ, μ as before.
Write $\lambda - \mu = \sum_i v_i \alpha_i$ and $\lambda = \sum_i w_i \omega_i$

$$V = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_i \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$$

where the first sum ranges over edges in the Dynkin quiver of G_Q. The Higgs branch $T^*V /\!\!/ G$ is a Nakajima quiver variety, where $G = \prod GL_{v_i}$.

Theorem (B-F-K-Kodera-N-W-W)

The Coulomb branch for this gauge theory is $\overline{\mathcal{W}}^{\lambda}_{\mu}$ and we have

$$\mathcal{A}(G, V) \cong Y^{\lambda}_{\mu}$$

Parabolic restriction for Coulomb branches

Fix G, V as before, and choose any $\xi : \mathbb{C}^\times \to T \subset G$. Let L_ξ be the centralizer of the image.
We would like to relate

$$\mathcal{A}(G, V) \text{ and } \mathcal{A}(L_\xi, V^\xi)$$

Theorem (KWWY)

There is a functor

$$\mathcal{A}(G, V)\text{-}GTmod \to \mathcal{A}(L_\xi, V^\xi)\text{-}GTmod$$

defined by a more complicated version of Hamiltonian reduction.

Application to affine Grassmannian slices

Choose G, V as above with $M_c(G,V) = \overline{\mathcal{W}}^{\lambda}_{\mu}$. Choose $\xi : \mathbb{C}^\times \to G = \prod GL_{v_i}$ using the first fundamental coweight for GL_{v_i}.

$$L_\xi = \prod_{j \neq i} GL_{v_j} \times GL_{v_i - 1} \times \mathbb{C}^\times$$

$$M_C(L_\xi, V^\xi) = \overline{\mathcal{W}}^{\lambda}_{\mu+\alpha_i} \times \overline{\mathcal{W}}^{0}_{-\alpha_i}$$

Theorem

This leads to a functor Y^λ_μ-GTmod $\to Y^\lambda_{\mu+\alpha_i}$-GTmod fitting into a commutative diagram

$$\begin{array}{ccc} Y^\lambda_\mu\text{-}GTmod & \xrightarrow{\sim} & T^\lambda_\mu\text{-}mod \\ \downarrow & & \downarrow E_i \\ Y^\lambda_{\mu+\alpha_i}\text{-}GTmod & \xrightarrow{\sim} & T^\lambda_{\mu+\alpha_i}\text{-}mod \end{array}$$