Shifted quantum affine algebras

monoidal categorification and Langlands duality

Conference Quantum groups and cohomology theory of quiver and flag varieties

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Two motivations

① Physics / Geometry

quantum field theory

$(G, M) \rightsquigarrow$ $\boxed{\text{SUper SYmmetric Gauge Theory } d=3}$

Complex reductive group

symplectic representation of G

(ω: symplectic form on M preserved by the action of G)

HIGGS branches | COULOMB branches

Complex algebraic symplectic varieties

Mathematical constructions :

Higgs branch $\leftrightarrow$ Coulomb branch

Quiver varieties — BRAVERMAN - FINKELBERG - NAKAJIMA 2016

" Symplectic duals "
(Braden- Licata- Proudfoot - Webster)

BFN Coulomb branches: remarkable new varieties,

include important examples: slices in affine Grassmannians

Deformation quantization: Coulomb branches come with a non-commutative product on

↗ Homology

↘ K-theory

quantum Coulomb branches (from convolution product)

Homology: related to shifted Yangians

BRUNDAN-KLESHCHEV

KAMNITZER-WEBSTER-WEEKES-YACOBI, NAKAJIMA-WEEKES

K-theory: related to shifted quantum affine algebras

FINKELBERG-TSYMBALIUK 2017

② Second motivation:

Physics / Representation Theory

Quantum Integrable Systems

Study of their spectra ⇝ representations of "q-oscillator algebra" $\mathcal{O}_q$ and "contracted quantum groups"

BAZHANOV - LUKYANOV - ZAMOLODCHIKOV (corresponds to sl_2)

General types: representations of BOREL quantum affine algebra $U_q(\hat{\mathfrak{g}})$

(H. - JIMBO and FRENKEL - H.)

⇒ BAXTER polynomials describe spectra of QIM 2015

The BOREL algebra $U_q(\hat{\mathfrak{b}}) \subset U_q(\hat{\mathfrak{g}})$ quantum affine

Relevant simple representations:

the action of $U_q(\hat{\mathfrak{b}})$ can not be extended to $U_q(\hat{\mathfrak{g}})$.

Replacement for $U_q(\hat{\mathfrak{g}})$?

First Hint (Huafeng ZHANG): certain examples:

prefundamental representations of shifted Yangians.

Motivations 1 and 2 $\rightsquigarrow$ develop representation theory of shifted quantum affine algebras.

(I) Shifted quantum affine algebras

$\mathfrak{g}$ fin. dim. simple Lie algebra $/\mathbb{C}$.

$\hat{\mathfrak{g}}$: affine KAC-MOODY algebra

= central extension of $\mathcal{L} \otimes \mathfrak{g}$ $\quad \mathcal{L} = \mathbb{C}[t^{\pm 1}]$.

$q \in \mathbb{C}^*$ (not root of 1) quantization parameter

$\mathcal{U}_q(\hat{\mathfrak{g}})$: quantum affine algebra
q- deformation of $\mathcal{U}(\hat{\mathfrak{g}})$
quantum group in the sense of DRINFELD-JIMBO.

HOPF algebra: duals, tensor products of representations well-defined.

Finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ intensively studied.

In particular $\nearrow$ geometry: quiver varieties.

$\searrow$ categorification: cluster algebras.

powerful methods.

Now: let $\mu \in \Lambda$ weight lattice of $\mathfrak{g}^L$

LANGLANDS dual Lie algebra

$\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$

Shifted quantum affine algebra

FINKELBERG - TSYMBALIUK

Idea of the definition of $U_q^\mu(\hat{\mathfrak{g}})$: same DRINFELD presentation as $U_q(\hat{\mathfrak{g}})$

$H_i \in \mathfrak{h} \subset \mathfrak{g} \rightsquigarrow H_i \otimes t^m \in \mathcal{L}\mathfrak{g}$, $m \in \mathbb{Z}$

CARTAN subalgebra

$1 \leq i \leq n$

$\quad = rk(\mathfrak{g})$

$\rightsquigarrow h_{i,m} \in U_q(\hat{\mathfrak{g}})$

CARTAN-DRINFELD elements

$q_i = q^{r_i}$

$$\phi_i^-(z) = \phi_{i,0}^- \exp\Big((q_i^{-1} - q_i) \sum_{r>0} h_{i,-r} z^{-r}\Big)$$

$\in U_q(\hat{\mathfrak{g}})[[z^{-1}]]$

$$\boxed{z^{\alpha_i(\mu)} \phi^-_{i,\alpha_i(\mu)}} \exp\Big((q_i^{-1} - q_i) \sum_{r>0} h_{i,-r} z^{-r}\Big)$$

$\in U_q^\mu(\hat{\mathfrak{g}})[[z^{-1}]]$

Rem: $\mathcal{U}_q^0(\hat{\mathfrak{g}})$ is (essentially) $\mathcal{U}_q(\hat{\mathfrak{g}})$.

Prop (H.20)

* $\mu \in -\Lambda^+$ $\quad \mathcal{U}_q^\mu(\hat{\mathfrak{g}}) \supset \mathcal{U}_q(\hat{\mathfrak{b}})$

* existence of evaluation morphisms

$$ev_\alpha : \mathcal{U}_q^{-w^\vee}(\hat{\mathfrak{sl}}_2) \to \mathcal{O}_q \qquad \alpha \in \mathbb{C}^\times$$

Representation prefundamental: action of $\mathcal{U}_q(\hat{\mathfrak{b}})$ extended to $\mathcal{U}_q^{-w^\vee}(\mathfrak{sl}_2)$.

Indication: relevant algebras from the point of view of Quantum Integrable Models.

II Representations of shifted quantum affine algebras

$\mu \in \Lambda$ Category $\mathcal{O}^\mu$ of representations of $U_q^\mu(\hat{\mathfrak{g}})$.

Defined by standard conditions of weight spaces

(finite dim. + cone condition).

Analog to

$\rightarrow$ category of representations of shifted Yangians.

$\rightarrow$ category $\hat{\mathcal{O}}$ of $U_q(\hat{\mathfrak{g}})$-modules

$\rightarrow$ category $\mathcal{O}$ of $U_q(\hat{\mathfrak{b}})$-modules.

Thm (H.-JIMBO, 2012) Simple $U_q(\hat{b})$-modules in the category $\mathcal{O}$ are parameterized by $(n = \mathrm{rk}\, \mathfrak{g})$

$$\Psi = (\Psi_1(z), \ldots, \Psi_n(z))$$

$\Psi_i(z) \in \mathbb{C}(z)$ regular at 0.

Thm 1 (H., 2020) Simple $U_q^\mu(\hat{\mathfrak{g}})$-modules in the category $\mathcal{O}^\mu$ are parameterized by

$$\Psi = (\Psi_1(z), \ldots, \Psi_n(z))$$

$\Psi_i(z) \in \mathbb{C}(z)$ regular at 0

$\deg(\Psi_i) = \alpha_i(\mu) \in \mathbb{Z}$.

Ex: for $1 \leq i \leq n$, $a \in \mathbb{C}^*$ $\Psi_{i,a} = (1, \ldots, 1, \underset{\text{position } i}{1 - za}, 1, \ldots, 1)$

$L(\Psi_{i,a})$ prefundamental representation of $U_q^{\omega_i^\vee}(\hat{\mathfrak{g}})$

$L(\Psi_{i,a}^{-1})$ ———————— of $U_q^{-\omega_i^\vee}(\hat{\mathfrak{g}})$.

Rem: representations of $U_q(\hat{\mathfrak{b}})$ / $U_q^\mu(\hat{\mathfrak{g}})$ different in general.

$L(\Psi_{i,a})$ of dim 1 as $U_q^{w_i^\vee}(\hat{\mathfrak{g}})$-mod

$L(\Psi_{i,a})$ of ∞ dim as $U_q(\hat{\mathfrak{b}})$-mod.

However: There are functors (H., 20)

induction $\quad \mathcal{O}_\mu \to \mathcal{O}$

restriction $\quad \mathcal{O} \to \mathcal{O}_\mu$

for $\mu \in \Lambda$.

from which one can get results.

Rem: if $\mu \in -\Lambda^+$, $L(\Psi)$ simple $U_q^\mu(\hat{\mathfrak{g}})$-module is also simple as a $U_q(\hat{\mathfrak{b}})$-module.

Thm 2 (H.20) • $U_q^\mu(\hat{\mathfrak{g}})$ has a non zero fin. dimensional representation $\Leftrightarrow$ $\mu \in \Lambda^+$.

• $L(\Psi)$ simple representation of $U_q^\mu(\hat{\mathfrak{g}})$

$$\dim(L(\Psi)) < +\infty \Leftrightarrow \Psi(z)\Psi(0)^{-1} \text{ product of radicals}$$

$$\underset{1 \leq i \leq n,\, a \in \mathbb{C}^*}{\Psi_{i,a}}, \quad \Psi_{i,aq_i^{-1}} / \Psi_{i,aq_i}.$$

Rem. • generalizes CHARI-PRESSLEY result for $U_q(\hat{\mathfrak{g}})$

• compatible with known result for ADE shifted Yangians

$$G^{sh} = \bigoplus_{\mu \in \Lambda} G^\mu \supset \bigoplus_{\mu \in \Lambda^+} G^\mu \supset \underset{\text{fin.dim. rep}^\varepsilon}{\mathcal{C}^{sh}}$$

abelian categories.

III Grothendieck ring and cluster algebras

There is a "coproduct" defined in a completion:

$$\Delta_{\mu,\mu'} : \mathcal{U}_q^{\mu+\mu'}(\hat{\mathfrak{g}}) \to \mathcal{U}_q^{\mu}(\hat{\mathfrak{g}}) \hat{\otimes} \mathcal{U}_q^{\mu'}(\hat{\mathfrak{g}})$$

for $\mu, \mu' \in \Lambda$ $\qquad \hat{\otimes}$: completed tensor product

(using DRINFELD coproduct formulas).

Can not be used to define directly tensor products of representations.

But, using deformation / renormalization procedure

$\rightsquigarrow$ ring structure on $\quad K(\mathcal{O}^{sh}) = \bigoplus_{\mu \in \Lambda} K(\mathcal{O}^{\mu})$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \cup$

no Chart $\qquad\qquad\qquad K(\mathcal{C}^{sh})$ is a subring.

<u>Reminder</u>: for Q a quiver $\rightsquigarrow$ $\mathcal{A}_Q$ <u>cluster algebra</u>.

(FOMIN – ZELEVINSKY). Subalgebra of a commutative fraction field $\mathbb{Q}(Y_1, \ldots, Y_N)$ generated by <u>cluster variables</u> obtained inductively from the <u>initial cluster variables</u> $Y_1, \ldots, Y_n$ by <u>mutations</u>.

H.–LECLERC: monoidal categorification of a cluster algebra $\mathcal{A}_Q$ by a monoidal category $\mathcal{M}$:

$$\mathcal{A}_Q \simeq K(\mathcal{M}) \quad \text{ring isomorphism}$$

so that cluster variables $\subset$ simple classes

Combining → induction / restriction functors with $U_q(\hat{b})$ - modules

→ H. - LECLERC monoidal categorifications

→ KASHIWARA - KIM - OH - PARK
recent proof of HL conjecture for quantum affine algebras

Thm 3 (H. 20) There is a cluster algebra structure

$$K(\mathcal{C}^{sh}) \simeq A_Q$$

no Chiral

- initial seed : prefund rep$^+$ of $U_q^{w_i^\vee}(\hat{g})$
- simple classes ⊃ cluster variables monomials

IV Truncations and Langlands duality

- $\underset{\Lambda^+}{\underset{\cap}{\nu}} \geq \mu$ and $Z = (Z_i(g))_{1 \leq i \leq n}$ polynomial so that $\deg(Z_i) = \nu(\alpha_i)$.

FINKELBERG-TSYMBALIUK: $\mathcal{U}^{\mu}_{q,\nu}(\hat{\mathfrak{g}})$

truncated shifted quantum affine algebra.

<u>Rem</u>:
- quotient of $\mathcal{U}^{\mu}_{q}(\hat{\mathfrak{g}})$
- follow definition truncated shifted Yangians
- λ, μ: analog to parameters for generalized slices W^{λ}_{μ} in affine Grassmannians.

Thm 4 (H. 20)

(i) $\mathcal{U}_{q,\nu}^{\mu}(\hat{\mathfrak{g}})$ has a finite number of simple modules (in $\mathcal{O}^{\mu}$).

(ii) $L(\Psi)$ such a simple module

$$\Rightarrow \quad \Psi \preceq Z$$

(a related partial ordering).

<u>Rem</u>: (i) already known for shifted Yangians by a different method.

<u>Question</u>: which simple module in $\mathcal{O}^{\mu}$ descend to $\mathcal{U}_{q,\nu}^{\mu}(\hat{\mathfrak{g}})$?

Solved for ADE shifted Yangians [KTWWY]

answer related to quiver varieties
(interpreted in terms of symplectic duality).

Motivated also by Langlands duality, for non simply-laced types:

Conj (H.20) There is a bijection

$$\left(\begin{array}{l} \text{Simple classes in} \\ \bigoplus_{\nu} G_{\nu}^{\sim} \\ \quad (\text{rep}^{t} \text{ of } U_{q,v}^{\nu}(\hat{\mathfrak{g}})) \end{array} \right) \overset{1:1}{\longleftrightarrow} \left(\begin{array}{l} \text{Monomials in } (q)\text{-character} \\ \text{of the standard module} \\ \text{of } U_q(\hat{\mathfrak{g}}^{L}) \\ (\text{tensor product of fundamental} \\ \text{representations of parameters} \\ \text{given by } z). \end{array} \right)$$

<u>explicitly</u> given by interpolating (q,ϵ)-characters.

Rem: * interpolating (q,ϵ)-characters (FRENKEL-H., 2011)

(inspired from deformed W-algebras of FRENKEL-RESHETIKHIN)

depends on q, ϵ, α

$$
\begin{tikzcd}
 & \chi_{q,\epsilon} \arrow[dl, "\epsilon=1,\ \alpha=1"'] \arrow[dr, "q=e^{\frac{2i\pi}{r}},\ \alpha=0"] \arrow[dd, "\epsilon=1,\ \alpha=0"] & \\
\mathrm{Rep}(U_q(\hat{g})) \arrow[rr, leftrightarrow, dashed] & & \mathrm{Rep}(U_\epsilon(\hat{g}^L)) \\
 & \text{parameters in the Conjecture.} &
\end{tikzcd}
$$

* NAKAJIMA-WEEKES for shifted Yangians (truncated)

simples for simply-laced $\Longleftrightarrow$ simples for non simply-laced

Possible relation with our conjecture.

Thm 5 (H. 20) L simple <u>fin. dim</u> representation of $U_q^\mu(\hat{\mathfrak{g}})$.

Then there are $\nu \geq \mu$, parameters Z as in the Conjecture, so that L descends to $U_{q,\nu}^\mu(\hat{\mathfrak{g}})$.

<u>Proof</u>: relies on Baxter polynomiality for quantum integrable systems (FRENKEL-H.).