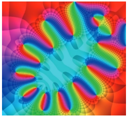
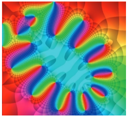
Geometric approach to Hitchin components via punctual Hilbert schemes

Alexander Thomas



Université de Strasbourg, Max-Planck Institute Bonn



joint with Vladimir Fock

8th october 2020

Teichmüller theory: classical, higher, super and quantum

Motivation

Goal

Describe Hitchin components in a geometric way, i.e. as the moduli space of some geometric structure.
In particular, we want to get rid of the fixed complex structure in Hitchin's parametrization.

- Teichmüller space $\mathcal{T}^2$ is moduli space of various geometric structures (hyperbolic, complex, conformal, ...)
- Hitchin components have representation-theoretic description
- Hitchin parametrization via Higgs bundle theory uses fixed complex structure on surface

We construct a new geometric structure, generalizing the complex structure, called **higher complex structure**.

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Non-abelian Hodge correspondence

Let (V, Φ) be a stable Higgs bundle. Then there is a unique (up to unitary gauge) flat connection D of the form

$$D = hd + \Phi + hA + h^2\Phi^*$$

where A is a unitary connection.
In coordinates: $\Phi = \Phi_1 dz, A = A_1 dz + A_2 d\bar{z}, \Phi^* = \Phi_1^* d\bar{z}$.

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The space of pairs of commuting matrices modulo conjugation is the **punctual Hilbert scheme of the plane**.

- Too many parameters, so we impose Φ_1 to be regular and nilpotent.

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Plan

1. Punctual Hilbert scheme
2. Higher complex structures
3. Further developments
   - $GL_2(\mathbb{R})$-action
   - "Higher" mapping class group?

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Punctual Hilbert scheme - Introduction

- Hilbert schemes exist in great generality: they describe all subschemes of a given scheme with some property.
- Example to have in mind: Grassmannian $Gr(k, n)$ is the set of all k-planes in $\mathbb{R}^n$.

Here, we consider the plane $\mathbb{C}^2 = \text{Spec } \mathbb{C}[x, y]$ and all zero-dimensional subschemes.

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The **punctual Hilbert scheme of the plane**, denoted by $\mathrm{Hilb}^n(\mathbb{C}^2)$, is the space of all ideals in $\mathbb{C}[x, y]$ of codimension n:

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \{I \text{ ideal in } \mathbb{C}[x,y] \mid \dim \mathbb{C}[x,y]/I = n\}.$$

The **zero-fiber** $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ consists of those ideals which are supported at the origin.

Theorem (Fogarty-Grothendieck, Haiman)

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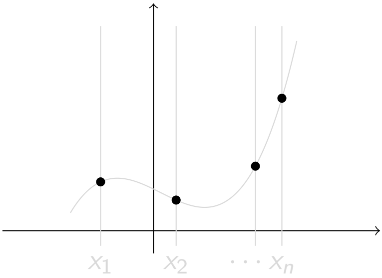
Punctual Hilbert scheme - Examples

Consider *n generic* points in $\mathbb{C}^2$.



There is a Lagrange interpolation polynomial $y = Q(x)$. Further we have $P(x) = \prod_i (x - x_i) = 0$ on our points. Thus:

$$I = \langle -x^n + t_1 x^{n-1} + t_2 x^{n-2} + \ldots + t_n, -y + \mu_1 + \mu_2 x + \ldots + \mu_n x^{n-1} \rangle.$$

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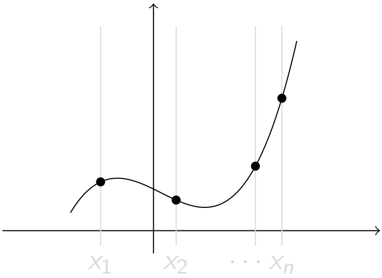

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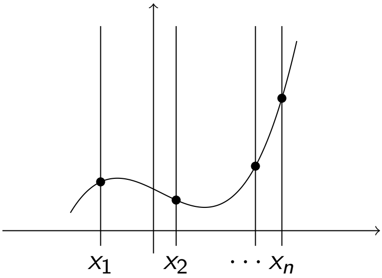

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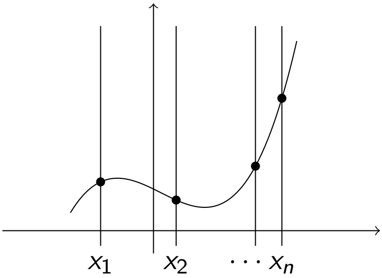

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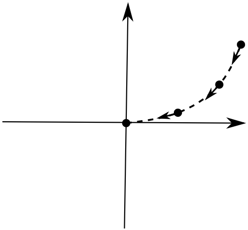

Roughly speaking: *the zero-fiber describes jets of curves at the origin.*

In particular:

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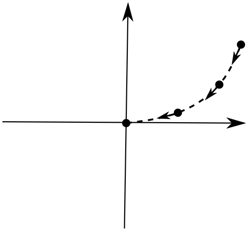

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Punctual Hilbert scheme - Matrix viewpoint

Given a pair of commuting matrices (A, B), you can *simultaneously trigonalize* them.

$$A \sim \begin{pmatrix} x_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & x_n \end{pmatrix} \text{ and } B \sim \begin{pmatrix} y_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & y_n \end{pmatrix}$$

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- $[M_x, M_y] = 0$
- $\mathbb{C}[x,y]/I$ is generated as $\mathbb{C}[x,y]$-module by $1 \in \mathbb{C}[x,y]/I$

Proposition

$\mathrm{Hilb}^n(\mathbb{C}^2) \cong \{(A,B) \in \mathfrak{gl}_n^2 \mid [A,B] = 0, (A,B) \textit{ admits cyclic vector}\} / \mathrm{GL}_n$

Converse direction:

$$(A,B) \mapsto I = \{P \in \mathbb{C}[x,y] \mid P(A,B) = 0\}$$

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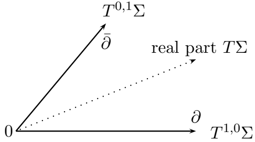
Σ: smooth closed surface of genus $g \geq 2$

$$\text{Complex structure} = \text{decomposition } T^{\mathbb{C}}\Sigma = T^{(1,0)}\Sigma \oplus T^{(0,1)}\Sigma$$



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(+ reality constraint)

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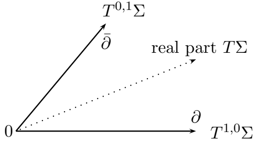

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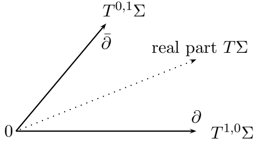

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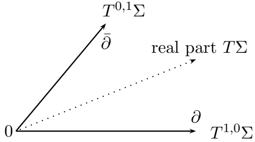

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A **higher complex structure** of order n on a surface Σ, in short **n-complex structure**, is a section I of $\mathrm{Hilb}^n_0(T^{*\mathbb{C}}\Sigma)$ such that at each point $z \in \Sigma$ the sum $I(z) + \bar{I}(z)$ is the maximal ideal supported at the origin of $T^{*\mathbb{C}}_z\Sigma$.

Notation:

- reference complex structure $(z, \bar{z})$ on Σ
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Locally we can write:

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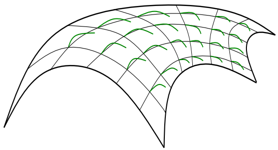

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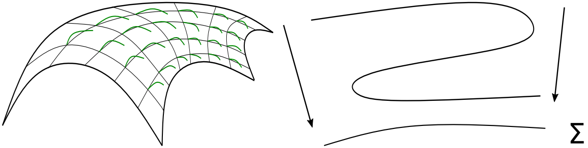

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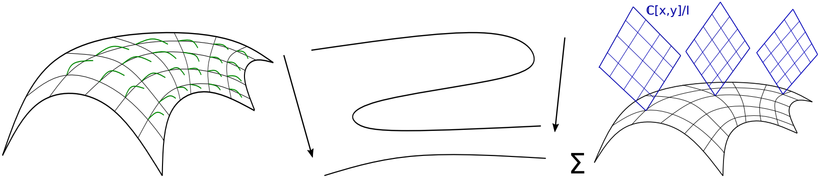
- as the collapse of a n-fold cover to the zero-section, or as the $(n-1)$-jet of a complex surface along the zero-section inside $T^{*\mathbb{C}}\Sigma$
- as a matrix-valued 1-form locally written as $\Phi_1(z,\bar{z})dz + \Phi_2(z,\bar{z})d\bar{z}$ with (Φ_1, Φ_2) commuting nilpotent matrices



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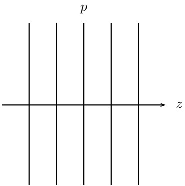
A **higher diffeomorphism** of a surface Σ is a hamiltonian diffeomorphism of $T^*\Sigma$ preserving the zero-section $\Sigma \subset T^*\Sigma$ setwise. The group of higher diffeomorphisms is denoted by $\mathrm{Ham}_\Sigma(T^*\Sigma)$.



- $\mathrm{Ham}_\Sigma(T^*\Sigma)$ acts on sections of $T^{*\mathbb{C}}\Sigma$, so on $\mathrm{Hilb}_0^n(T^{*\mathbb{C}}\Sigma)$
- action on generators: $I = \langle P, Q \rangle$
  Hamiltonian H (function on $T^*\Sigma$) acts by

$$\delta P = \{H, P\} \mod I$$
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Theorem (Fock, T., 2016)

Any two higher complex structures are locally equivalent under higher diffeomorphisms.

Higher complex structures - Local theory

- $\text{Ham}_\Sigma(T^*\Sigma)$ acts on sections of $T^{*\mathbb{C}}\Sigma$, so on $\text{Hilb}_0^n(T^{*\mathbb{C}}\Sigma)$
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Definition

The **moduli space of higher complex structures**, denoted by $\hat{\mathcal{T}}^{\mathbf{n}}$, is the space of all n-complex structures modulo higher diffeomorphisms.

Theorem (Fock, T., 2016)

The moduli space $\hat{\mathcal{T}}^n$ has the following properties:

- *Contractible manifold of complex dimension $(n^2-1)(g-1)$,*
- *Forgetful map: $\hat{\mathcal{T}}^n \to \hat{\mathcal{T}}^{n-1}$,*
- *Copy of Teichmüller space: $\mathcal{T}^2 \hookrightarrow \hat{\mathcal{T}}^n$,*
- *Cotangent space for $\mu \in \mathcal{T}^2$: $T^*_\mu \hat{\mathcal{T}}^n = \bigoplus_{m=2}^n H^0(K^m)$,*
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Conjecture

Our moduli space $\hat{\mathcal{T}}^n$ is canonically diffeomorphic to Hitchin's component.

Hitchin component	$\hat{\mathcal{T}}^n$
contractible	contractible
dimension $(n^2-1)(2g-2)$	dimension $(n^2-1)(2g-2)$
$MCG(\Sigma)$-action	$MCG(\Sigma)$-action
complex structure?	complex structure
symplectic structure	symplectic structure?
Kähler structure?	Kähler structure?
cotangent bundle?	explicit description of $T^*\hat{\mathcal{T}}^n$
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How to compute $T^*\hat{\mathcal{T}}^n$?

- punctual Hilbert scheme $\text{Hilb}^n(\mathbb{C}^2)$ is symplectic
- zero-fiber $\text{Hilb}^n_0(\mathbb{C}^2)$ is Lagrangian in the *reduced Hilbert scheme* $\text{Hilb}^n_{red}(\mathbb{C}^2)$
  - generically, n points with barycenter 0
  - pairs of commuting matrices in $\mathfrak{sl}_n$
  - typical ideal is of the form

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Thus, $T^* \text{Hilb}^n_0(\mathbb{C}^2) = T^{normal} \text{Hilb}^n_0(\mathbb{C}^2) \approx \text{Hilb}^n_{red}(\mathbb{C}^2)$.

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Theorem (Fock, T., 2017)

$$T^*\hat{\mathcal{T}}^n = \Big\{[(\mu_2, ..., \mu_n, t_2, ..., t_n)] \mid \mu_k \in \Gamma(K^{1-k} \otimes \bar{K}), t_k \in \Gamma(K^k) \text{ and } \forall k$$

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Proof structure for main conjecture

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Deform coordinates (μ_i, t_i) of $T^*\hat{\mathcal{T}}^n$ into $(\hat{\mu}_i(h), \hat{t}_i(h))$ with

$$\hat{\mu}_i(h) = \mu_i + \mathcal{O}(h) \quad \text{and} \quad \hat{t}_i(h) = ht_i + \mathcal{O}(h^2)$$

such that we get a *flat* connection of the form

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$\mathrm{GL}_2(\mathbb{R})$−action

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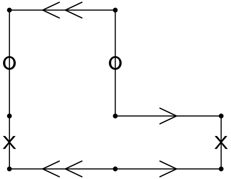
$$z \mapsto \int_{z_0}^{z} \sqrt{t}.$$



Natural $\mathrm{GL}_2(\mathbb{R})$−action on space of polygons, so on $T^*\mathcal{T}^2$.

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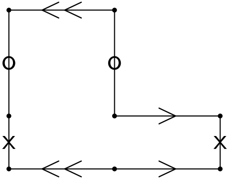

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Proposition

There is a $\mathrm{GL}_2(\mathbb{R})-$*action on* $T^*\hat{\mathcal{T}}^n$.

Recall that the coordinates on $T^*\hat{\mathcal{T}}^n$ are given by

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$$\mathrm{GL}_2(\mathbb{R}) \cong \left\{ \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} \mid a\bar{a} - b\bar{b} \neq 0 \right\}$$

A matrix $\left(\begin{smallmatrix} a & \bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$ induces the action:

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Based on work of Ono and Banyaga, we get:

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Open question

Is this quotient group strictly bigger than $\mathrm{MCG}(\Sigma)$?

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Open question

Is this quotient group strictly bigger than $\mathrm{MCG}(\Sigma)$?

Perspectives

- Generalization to Lie algebras $\mathfrak{g}$ other than $\mathfrak{sl}_n$:
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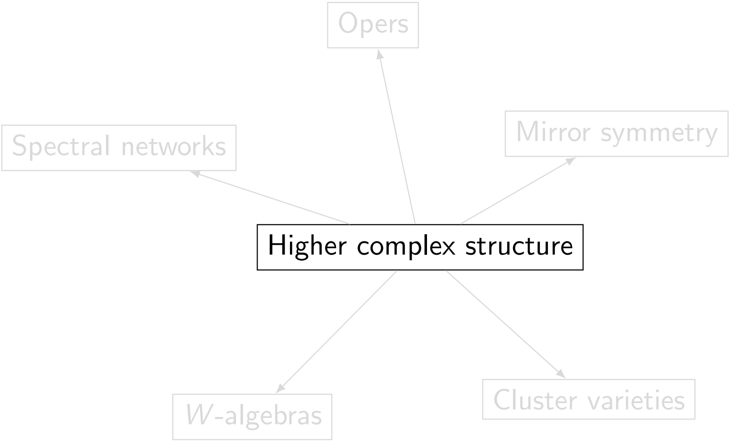

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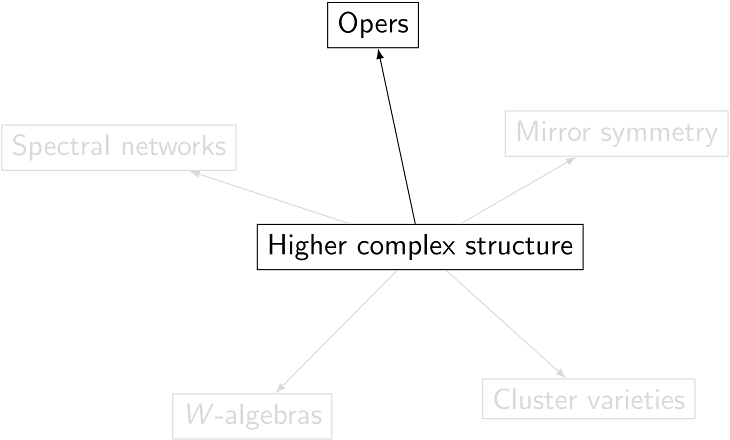

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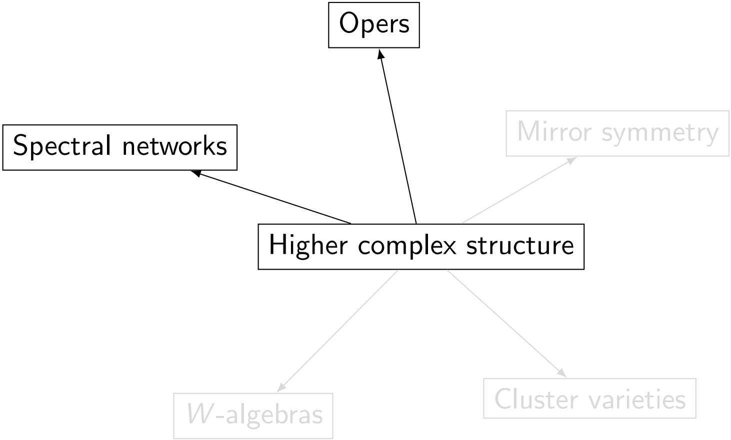

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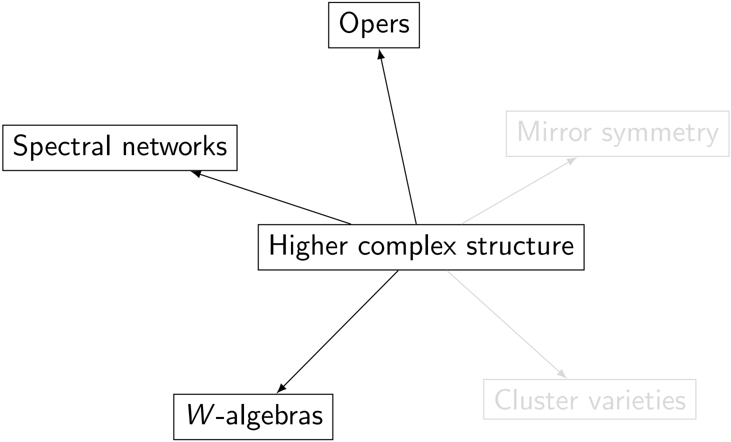

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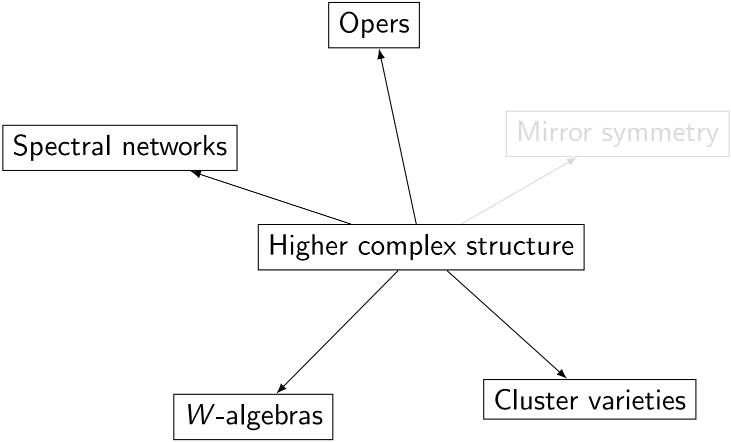

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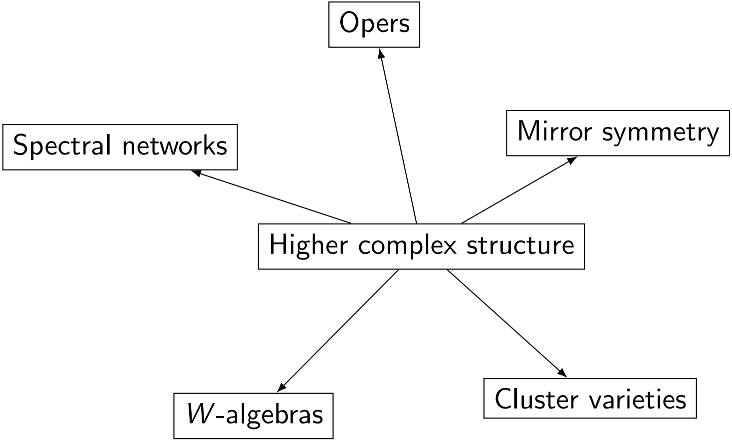
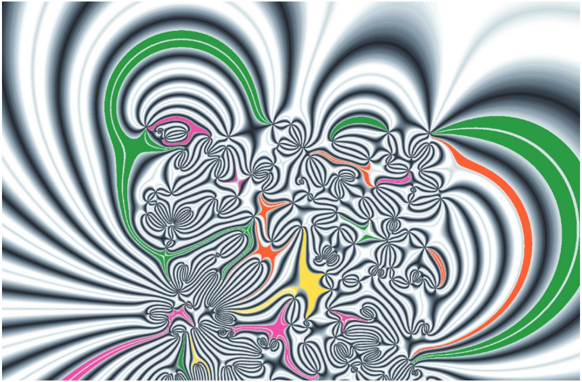


Thank you for your attention!



The way to understanding is long and wiggly...

Image sources: maths calendar "Complex beauties". Just click on the images!