Teichmüller flow and complex geometry of Moduli Spaces.

Vlad Markovic

Autumn 2020

Intrinsic geometry of complex manifolds

- The four invariant (pseudo) metrics are: Kähler-Einstein, Bergman, Kobayashi, and Carathéodory metrics. The goal is to study these metrics on and relate them to each other.

Intrinsic geometry of complex manifolds

- The four invariant (pseudo) metrics are: Kähler-Einstein, Bergman, Kobayashi, and Carathéodory metrics. The goal is to study these metrics on and relate them to each other.
- Among them, Kobayashi and Carathéodory (pseudo) metrics are naturally defined on all complex manifolds.

Intrinsic geometry of complex manifolds

- The four invariant (pseudo) metrics are: Kähler-Einstein, Bergman, Kobayashi, and Carathéodory metrics. The goal is to study these metrics on and relate them to each other.
- Among them, Kobayashi and Carathéodory (pseudo) metrics are naturally defined on all complex manifolds.
- These two metrics are non-expanding under holomorphic maps and every other holomorphically natural metric is sandwiched between the two.

Intrinsic geometry of complex manifolds

- Kobayashi metric d_K dominates Carathéodory metric d_C on every complex manifold. The question is when they agree.

Intrinsic geometry of complex manifolds

- Kobayashi metric d_K dominates Carathéodory metric d_C on every complex manifold. The question is when they agree.

Theorem (Kobayashi)

The identity $d_K \equiv d_C$ holds on every symmetric hermitian space.

Intrinsic geometry of complex manifolds

- Kobayashi metric d_K dominates Carathéodory metric d_C on every complex manifold. The question is when they agree.

Theorem (Kobayashi)

The identity $d_K \equiv d_C$ holds on every symmetric hermitian space.

Theorem (Lampert)

The identity $d_K \equiv d_C$ holds on every convex domain in $\mathbb{C}^n$.

Intrinsic geometry of negatively curved Kähler manifolds

- The interplay between the invariant metrics is particularly rich when the underlying Kähler manifold has negative sectional curvature.

Intrinsic geometry of negatively curved Kähler manifolds

- The interplay between the invariant metrics is particularly rich when the underlying Kähler manifold has negative sectional curvature.

Theorem (Wu-Yau)

On a complete Kähler manifold, whose sectional curvatures are pinched between two negative constants, the three classical invariant metrics, Kähler-Einstein, Bergman, and Kobayashi metrics exist and are all quasi-isometric to each other.

Intrinsic geometry of negatively curved Kähler manifolds

- The major open problem in this field is to show that the Carathéodory metric is non-degenerate.

Intrinsic geometry of negatively curved Kähler manifolds

- The major open problem in this field is to show that the Carathéodory metric is non-degenerate.

Conjecture

On a complete Kähler manifold X, whose sectional curvatures are pinched between two negative constants, the Carathéodory metric exists and is quasi-isometric to other invariant metrics.

Intrinsic geometry of negatively curved Kähler manifolds

- The major open problem in this field is to show that the Carathéodory metric is non-degenerate.

Conjecture

On a complete Kähler manifold X, whose sectional curvatures are pinched between two negative constants, the Carathéodory metric exists and is quasi-isometric to other invariant metrics.

- For example, the conjecture holds when X is the universal cover of a compact Kähler manifold which maps non-trivially to a closed surface of genus at least two.

Ambient geometry of negatively curved Kähler manifolds

- The only known examples of negatively curved, complete, simply connected Kähler manifolds arise as bounded domains in $\mathbb{C}^n$.

Ambient geometry of negatively curved Kähler manifolds

- The only known examples of negatively curved, complete, simply connected Kähler manifolds arise as bounded domains in $\mathbb{C}^n$.

Conjecture

Every negatively curved, complete, and simply connected Kähler manifold is biholomorphic to a bounded domain in $\mathbb{C}^n$.

Ambient geometry of negatively curved Kähler manifolds

- The only known examples of negatively curved, complete, simply connected Kähler manifolds arise as bounded domains in $\mathbb{C}^n$.

Conjecture

Every negatively curved, complete, and simply connected Kähler manifold is biholomorphic to a bounded domain in $\mathbb{C}^n$.

- A weaker version of the previous conjecture is whether such Kähler manifolds admit bounded holomorphic functions (Yau in 1978).

Ambient geometry of negatively curved Kähler manifolds

- The only known examples of negatively curved, complete, simply connected Kähler manifolds arise as bounded domains in $\mathbb{C}^n$.

Conjecture

Every negatively curved, complete, and simply connected Kähler manifold is biholomorphic to a bounded domain in $\mathbb{C}^n$.

- A weaker version of the previous conjecture is whether such Kähler manifolds admit bounded holomorphic functions (Yau in 1978).
- Moreover, the question is what can be said about the regularity of such a domain in $\mathbb{C}^n$ (pseudo-convex, convex, smooth boundary, etc.).

The Teichmüller space

- Let $\mathcal{T}_{g,n}$ denote the Teichmüller space of marked genus g Riemann surfaces with n cusps.

The Teichmüller space

- Let $\mathcal{T}_{g,n}$ denote the Teichmüller space of marked genus g Riemann surfaces with n cusps.
- Teichmüller spaces are some of the main sources of examples of negatively curved (although non-complete), simply connected Kähler manifolds, and their properties motivate most of the above stated conjectures.

The Teichmüller space

- Let $\mathcal{T}_{g,n}$ denote the Teichmüller space of marked genus g Riemann surfaces with n cusps.
- Teichmüller spaces are some of the main sources of examples of negatively curved (although non-complete), simply connected Kähler manifolds, and their properties motivate most of the above stated conjectures.

Theorem (Bers)

$\mathcal{T}_{g,n}$ is biholomorphic to a bounded domain in $\mathbb{C}^{3g-3+n}$.

The Teichmüller space

- The main question in studying the ambient geometry of Teichmüller spaces is what is the optimal regularity of such bounded domains.

The Teichmüller space

- The main question in studying the ambient geometry of Teichmüller spaces is what is the optimal regularity of such bounded domains.
- It is expected that any biholomorphic embedding of $\mathcal{T}_{g,n}$ into $\mathbb{C}^{3g-3+n}$ must be highly irregular.

The Teichmüller space

- The main question in studying the ambient geometry of Teichmüller spaces is what is the optimal regularity of such bounded domains.
- It is expected that any biholomorphic embedding of $\mathcal{T}_{g,n}$ into $\mathbb{C}^{3g-3+n}$ must be highly irregular.

Conjecture (Siu)

$\mathcal{T}_{g,n}$ is NOT biholomorphic to a convex domain in $\mathbb{C}^{3g-3+n}$.

Intrinsic geometry of $\mathcal{T}_{g,n}$

Theorem (Yau)

The Kobayashi and Carathèodory metrics are proportional, that is

$$q\, d_{\mathcal{T}}(p,q) \leq d_C(p,q) \leq d_{\mathcal{T}}(p,q),$$

for a constant $0 < q \leq 1$*.*

Intrinsic geometry of $\mathcal{T}_{g,n}$

Theorem (Yau)

The Kobayashi and Caratheodory metrics are proportional, that is

$$q\, d_{\mathcal{T}}(p,q) \leq d_C(p,q) \leq d_{\mathcal{T}}(p,q),$$

for a constant $0 < q \leq 1$.

- One of the most important results in Teichmüller theory is the theorem of Royden that $d_{\mathcal{T}} \equiv d_K$. Since then, the following question remains:

Intrinsic geometry of $\mathcal{T}_{g,n}$

Theorem (Yau)

The Kobayashi and Carathèodory metrics are proportional, that is

$$q\, d_{\mathcal{T}}(p,q) \leq d_C(p,q) \leq d_{\mathcal{T}}(p,q),$$

for a constant $0 < q \leq 1$.

- One of the most important results in Teichmüller theory is the theorem of Royden that $d_{\mathcal{T}} \equiv d_K$. Since then, the following question remains:

Problem

Do the identities $d_{\mathcal{T}} \equiv d_K \equiv d_C$ *hold on every* $\mathcal{T}_{g,n}$ *?*

Teichmüller Discs

- Isometries of the hyperbolic plane $\mathbb{H}$ into $\mathcal{T}_{g,n}$ (with respect to $d_{\mathcal{T}}$) are called Teichmüller discs. Every two points in $\mathcal{T}_{g,n}$ lie on the unique Teichmüller disc (complex geodesic).

Teichmüller Discs

- Isometries of the hyperbolic plane $\mathbb{H}$ into $\mathcal{T}_{g,n}$ (with respect to $d_{\mathcal{T}}$) are called Teichmüller discs. Every two points in $\mathcal{T}_{g,n}$ lie on the unique Teichmüller disc (complex geodesic).
- The identity $d_C \equiv d_{\mathcal{T}}$ holds on a Teichmüller disc if and only $\mathcal{T}_{g,n}$ holomorphically retracts onto this disc.

Teichmüller Discs

- Isometries of the hyperbolic plane $\mathbb{H}$ into $\mathcal{T}_{g,n}$ (with respect to $d_{\mathcal{T}}$) are called Teichmüller discs. Every two points in $\mathcal{T}_{g,n}$ lie on the unique Teichmüller disc (complex geodesic).
- The identity $d_C \equiv d_{\mathcal{T}}$ holds on a Teichmüller disc if and only $\mathcal{T}_{g,n}$ holomorphically retracts onto this disc.

Lemma

The identities $d_{\mathcal{T}} \equiv d_K \equiv d_C$ hold if and only if $\mathcal{T}_{g,n}$ holomorphically retracts onto each Teichmüller disc.

Abelian Teichmüller Discs

The following theorem of Kra (rediscovered by McMullen) shows that $d_C \equiv d_{\mathcal{T}}$ on Abelian discs.

Abelian Teichmüller Discs

The following theorem of Kra (rediscovered by McMullen) shows that $d_C \equiv d_{\mathcal{T}}$ on Abelian discs.

Theorem (Kra,McMullen)

$\mathcal{T}_g$ holomorphically retracts onto each Teichmüller disc which is determined by a holomorphic quadratic differential with even order zeroes (this is satisfied when the differential is the square of an Abelian 1-form).

The Period Matrix

- Choose α_j, β_j, $j = 1, ..., g$, to be a symplectic basis for $H_1(S, \mathbb{Z})$, where $S \in \mathcal{T}_g$.

The Period Matrix

- Choose α_j, β_j, $j = 1, ..., g$, to be a symplectic basis for $H_1(S, \mathbb{Z})$, where $S \in \mathcal{T}_g$.
- We select 1-forms ω_i on S such that

$$\int_{\alpha_j} \omega_i = \delta_{ij}.$$

The Period Matrix

- Choose α_j, β_j, $j = 1, ..., g$, to be a symplectic basis for $H_1(S, \mathbb{Z})$, where $S \in \mathcal{T}_g$.
- We select 1-forms ω_i on S such that

$$\int_{\alpha_j} \omega_i = \delta_{ij}.$$

- Let Σ_g denote the Siegel upper half-space of $g \times g$ complex matrices with positive-definite imaginary part.

The Period Matrix

- Define $\sigma : \mathcal{T}_g \to \Sigma_g$ by

$$\sigma_{ij}(S) = \int_{\beta_j} \omega.$$

The Period Matrix

- Define $\sigma : \mathcal{T}_g \to \Sigma_g$ by

$$\sigma_{ij}(S) = \int_{\beta_j} \omega.$$

- Kra and McMullen prove that every Abelian Teichmüller disc τ^{φ} is mapped by σ onto a complex geodesic in Σ_g.

The Period Matrix

- Define $\sigma : \mathcal{T}_g \to \Sigma_g$ by
$$\sigma_{ij}(S) = \int_{\beta_j} \omega.$$

- Kra and McMullen prove that every Abelian Teichmüller disc τ^φ is mapped by σ onto a complex geodesic in Σ_g.
- Thus
$$d_C \equiv d_{\mathcal{T}}$$
on τ^φ because Σ_g is a Hermitian symmetric space.

The Universal Teichmüller Space

- The Universal Teichmüller space $\mathcal{T}(\mathbb{D})$ is the space of univalent holomorphic functions $f : \mathbb{D}^* \to \mathbb{C}^*$ that have a quasiconformal extension to $\mathbb{D}$.

The Universal Teichmüller Space

- The Universal Teichmüller space $\mathcal{T}(\mathbb{D})$ is the space of univalent holomorphic functions $f : \mathbb{D}^* \to \mathbb{C}^*$ that have a quasiconformal extension to $\mathbb{D}$.
- Grunsky's coefficients α_{mn} are determined by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = -\sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}.$$

- The classical Grunsky theorem states that a holomorphic function $f : \mathbb{D}^* \to \mathbb{C}$ is univalent if and only if

$$\chi(f) = \sup_{(x_n)} \left| \sum_{m,n} \sqrt{mn}\, \alpha_{mn} x_m x_n \right| \leq 1,$$

where (x_n) varies over all sequences in l^2 of norm 1.

The Universal Teichmüller Space

- The classical Grunsky theorem states that a holomorphic function $f : \mathbb{D}^* \to \mathbb{C}$ is univalent if and only if

$$\chi(f) = \sup_{(x_n)} \left| \sum_{m,n} \sqrt{mn}\, \alpha_{mn} x_m x_n \right| \leq 1,$$

  where (x_n) varies over all sequences in l^2 of norm 1.

- The connection with Teichmüller norm is encapsulated in the inequalities

$$\chi(f) \leq k(f) \leq \sqrt{\frac{9}{8}} \chi(f),$$

  where $k(f)$ is the minimal possible dilatation of the quasiconformal extension of f.

- Moreover, the equality $\chi(f) = k(f)$ holds if and only f belongs to the Teichmüller disc corresponding to a quadratic differential with no odd order zeroes. Using this Kruskhal proved that $d_C = d_{\mathcal{T}}$ on such discs in $\mathcal{T}(\mathbb{D})$.

The Universal Teichmüller Space

- Moreover, the equality $\chi(f) = k(f)$ holds if and only f belongs to the Teichmüller disc corresponding to a quadratic differential with no odd order zeroes. Using this Kruskhal proved that $d_C = d_{\mathcal{T}}$ on such discs in $\mathcal{T}(\mathbb{D})$.
- To make things interesting, Shiga and Tanigawa found examples of Teichmüller discs in $\mathcal{T}(\mathbb{D})$ where $d_C = d_{\mathcal{T}}$ that correspond to quadratic differentials containing odd order zeroes.

Main Results

Theorem (Markovic)

$\mathcal{T}_{0,5}$ does not holomorphically retract onto a Teichmüller disc.

Main Results

Theorem (Markovic)

$\mathcal{T}_{0,5}$ does not holomorphically retract onto a Teichmüller disc.

Combining this with the fact that $\mathcal{T}_{0,5}$ embeds isometrically into each $\mathcal{T}_g$ yields:

Main Results

Theorem (Markovic)

$\mathcal{T}_{0,5}$ does not holomorphically retract onto a Teichmüller disc.

Combining this with the fact that $\mathcal{T}_{0,5}$ embeds isometrically into each $\mathcal{T}_g$ yields:

Theorem (Markovic)

$\mathcal{T}_g$ does not holomorphically retract onto at least one Teichmüller disc. In particular,

$$d_C \neq d_{\mathcal{T}}$$

on $\mathcal{T}_g$, for every $g \geq 2$.

The conjecture of Siu

- In the 80's Siu conjectured that $\mathcal{T}_g$ is not biholomorphic to a convex domain in $\mathbb{C}^{3g-3}$. The previous theorem yields this conjecture together with the Lempert's theorem stated at the beginning yields the solution to the Siu's conjecture.

The conjecture of Siu

- In the 80's Siu conjectured that $\mathcal{T}_g$ is not biholomorphic to a convex domain in $\mathbb{C}^{3g-3}$. The previous theorem yields this conjecture together with the Lempert's theorem stated at the beginning yields the solution to the Siu's conjecture.

Theorem (Markovic)

$\mathcal{T}_g$ is not biholomorphic to a convex domain in $\mathbb{C}^{3g-3}$.

Main Results

- If $\mathcal{T}_g$ holomorphically retracts onto every Teichmüller discs from a certain collection of such discs, then $\mathcal{T}_g$ holomorphically retracts onto every Teichmüller discs in the closure of this collection.

Main Results

- If $\mathcal{T}_g$ holomorphically retracts onto every Teichmüller discs from a certain collection of such discs, then $\mathcal{T}_g$ holomorphically retracts onto every Teichmüller discs in the closure of this collection.

Theorem

$\mathcal{T}_g$ does not holomorphically retract onto a random Teichmüller disc.

Classification of holomorphic retracts

- Each Teichmüller disc $\tau^{\varphi} : \mathbb{H} \to \mathcal{T}_g$ is determined by a holomorphic quadratic differential φ from the cotangent bundle of $\mathcal{T}_g$.

Classification of holomorphic retracts

- Each Teichmüller disc $\tau^{\varphi} : \mathbb{H} \to \mathcal{T}_g$ is determined by a holomorphic quadratic differential φ from the cotangent bundle of $\mathcal{T}_g$.

Conjecture

$\mathcal{T}_g$ holomorphically retracts onto a Teichmüller disc τ^{φ} if and only if all zeroes of the φ are of even order.

Topological definition of the Carathéodory metric

- Recall the Period Map $\sigma_g : \mathcal{T}_g \to \Sigma_g$.

Topological definition of the Carathéodory metric

- Recall the Period Map $\sigma_g : \mathcal{T}_g \to \Sigma_g$.
- Consider the isometric embedding $\iota_h : \mathcal{T}_g \to \mathcal{T}_h$, where $2h - 2 = d(2g - 2)$.

Topological definition of the Carathéodory metric

- Recall the Period Map $\sigma_g : \mathcal{T}_g \to \Sigma_g$.
- Consider the isometric embedding $\iota_h : \mathcal{T}_g \to \mathcal{T}_h$, where $2h - 2 = d(2g - 2)$.
- Let $\theta_h = \iota_h \circ \sigma_g : \mathcal{T}_g \to \Sigma_h$ be the induced holomorphic embedding.

Topological definition of the Carathéodory metric

- Recall the Period Map $\sigma_g : \mathcal{T}_g \to \Sigma_g$.
- Consider the isometric embedding $\iota_h : \mathcal{T}_g \to \mathcal{T}_h$, where $2h - 2 = d(2g - 2)$.
- Let $\theta_h = \iota_h \circ \sigma_g : \mathcal{T}_g \to \Sigma_h$ be the induced holomorphic embedding.

Conjecture

The Carathéodory metric d_C on $\mathcal{T}_g$ is computed as

$$d_C(p, q) = \sup_h d_{\Sigma_h}\big(\theta_h(p), \theta_h(q)\big).$$

The GL$(2, \mathbb{R})$ Action

- GL$(2, \mathbb{R})$ acts naturally on the space of holomorphic quadratic differentials $\mathcal{QD}_{g,n}$ which is the cotangent bundle over $\mathcal{T}_{g,n}$.

The $\mathrm{GL}(2,\mathbb{R})$ Action

- $\mathrm{GL}(2,\mathbb{R})$ acts naturally on the space of holomorphic quadratic differentials $\mathcal{QD}_{g,n}$ which is the cotangent bundle over $\mathcal{T}_{g,n}$.
- Given $\varphi \in \mathcal{QD}_{g,n}$, the Teichmüller disc τ^{φ} is the image of the orbit $\mathrm{GL}(2,\mathbb{R})\big(\varphi\big)$ under the projection $\mathcal{QD}_{g,n} \to \mathcal{T}_{g,n}$.

The $\mathrm{GL}(2,\mathbb{R})$ Action

- $\mathrm{GL}(2,\mathbb{R})$ acts naturally on the space of holomorphic quadratic differentials $\mathcal{QD}_{g,n}$ which is the cotangent bundle over $\mathcal{T}_{g,n}$.
- Given $\varphi \in \mathcal{QD}_{g,n}$, the Teichmüller disc τ^{φ} is the image of the orbit $\mathrm{GL}(2,\mathbb{R})(\varphi)$ under the projection $\mathcal{QD}_{g,n} \to \mathcal{T}_{g,n}$.

Proposition

If $\mathcal{T}_{g,n}$ holomorphically retracts onto a Teichmüller disc then the same holds for every Teichmüller disc in the closure of its $\mathrm{GL}(2,\mathbb{R})$ *orbit in the Moduli Space $\mathcal{M}_{g,n}$.*

The $\mathrm{GL}(2, \mathbb{R})$ Orbit Closure

- So, to show that $\mathcal{T}_{g,n}$ does not holomorphically retracts onto a Teichmüller disc, it is enough to establish this for any disc in its $\mathrm{GL}(2, \mathbb{R})$ orbit closure in the Moduli Space $\mathcal{M}_{g,n}$.

The $\mathrm{GL}(2,\mathbb{R})$ Orbit Closure

- So, to show that $\mathcal{T}_{g,n}$ does not holomorphically retracts onto a Teichmüller disc, it is enough to establish this for any disc in its $\mathrm{GL}(2,\mathbb{R})$ orbit closure in the Moduli Space $\mathcal{M}_{g,n}$.
- We establish a criterion when $\mathcal{T}_{g,n}$ holomorphically retracts onto $\tau^{\varphi} : \mathbb{H} \to \mathcal{T}_{g,n}$ when φ is a Jenkins-Strebel differentials.

The $\mathrm{GL}(2,\mathbb{R})$ Orbit Closure

- So, to show that $\mathcal{T}_{g,n}$ does not holomorphically retracts onto a Teichmüller disc, it is enough to establish this for any disc in its $\mathrm{GL}(2,\mathbb{R})$ orbit closure in the Moduli Space $\mathcal{M}_{g,n}$.
- We establish a criterion when $\mathcal{T}_{g,n}$ holomorphically retracts onto $\tau^{\varphi} : \mathbb{H} \to \mathcal{T}_{g,n}$ when φ is a Jenkins-Strebel differentials.
- The criterion is vacuous for Jenkins-Strebel differentials φ with a single cylinder.

The $\mathrm{GL}(2,\mathbb{R})$ Orbit Closure

- So, to show that $\mathcal{T}_{g,n}$ does not holomorphically retracts onto a Teichmüller disc, it is enough to establish this for any disc in its $\mathrm{GL}(2,\mathbb{R})$ orbit closure in the Moduli Space $\mathcal{M}_{g,n}$.
- We establish a criterion when $\mathcal{T}_{g,n}$ holomorphically retracts onto $\tau^{\varphi} : \mathbb{H} \to \mathcal{T}_{g,n}$ when φ is a Jenkins-Strebel differentials.
- The criterion is vacuous for Jenkins-Strebel differentials φ with a single cylinder.
- Heuristically speaking, the more cylinders φ has the more effective the criterion becomes.

Main Results: The Second Generation

- Combined work of Minsky-Weiss and Smillie-Weiss shows that the $\mathrm{GL}(2, \mathbb{R})$ orbit closure of any holomorphic quadratic differential contains a Jenkins-Strebel differential.

Main Results: The Second Generation

- Combined work of Minsky-Weiss and Smillie-Weiss shows that the $\mathrm{GL}(2,\mathbb{R})$ orbit closure of any holomorphic quadratic differential contains a Jenkins-Strebel differential.
- Combining several major results in Teichmüller dynamics Wright shows that the $\mathrm{GL}(2,\mathbb{R})$ orbit closure of any holomorphic quadratic differential that has at least one odd order zero contains a Jenkins-Strebel differentials with more than one cylinder.

Main Results: The Second Generation

- Combined work of Minsky-Weiss and Smillie-Weiss shows that the $\mathrm{GL}(2,\mathbb{R})$ orbit closure of any holomorphic quadratic differential contains a Jenkins-Strebel differential.
- Combining several major results in Teichmüller dynamics Wright shows that the $\mathrm{GL}(2,\mathbb{R})$ orbit closure of any holomorphic quadratic differential that has at least one odd order zero contains a Jenkins-Strebel differentials with more than one cylinder.

Theorem (Gekhtman-Markovic)

$\mathcal{T}_{1,2}$ holomorphically retracts onto a Teichmüller disc τ^{φ} if and only if φ has zeroes of even order.

Teichmüller Discs

- The Teichmüller disc $\tau^{\varphi} : \mathbb{H} \to \mathcal{T}_{g,n}$ is determined by the Beltrami dilatations

$$\lambda \to \left(\frac{\mathbf{i} - \lambda}{\mathbf{i} + \lambda} \right) \frac{|\varphi|}{\varphi}.$$

Teichmüller Discs

- The Teichmüller disc $\tau^{\varphi} : \mathbb{H} \to \mathcal{T}_{g,n}$ is determined by the Beltrami dilatations

$$\lambda \to \left(\frac{\mathbf{i} - \lambda}{\mathbf{i} + \lambda}\right) \frac{|\varphi|}{\varphi}.$$

- In terms of the $\mathrm{GL}(2, \mathbb{R})$ action, $\tau^{\varphi}(\lambda) = A(\varphi)$ where $\lambda = s + \mathbf{i}e^{t}$ and

$$A = \begin{bmatrix} e^{t} & s \\ 0 & e^{-t} \end{bmatrix}.$$

Teichmüller Polydics

- Assume $\varphi \in \mathcal{QD}(S)$ is a Jenkins-Strebel differential with the annuli $\Pi_1, ..., \Pi_k \subset S$ that are swept by closed horizontal trajectories of φ .

Teichmüller Polydics

- Assume $\varphi \in \mathcal{QD}(S)$ is a Jenkins-Strebel differential with the annuli $\Pi_1, ..., \Pi_k \subset S$ that are swept by closed horizontal trajectories of φ .
- We define the (Teichmüller) poly-plane $\mathcal{E} : \mathbb{H}^k \to \mathcal{T}_{g,n}$ by prescribing the Beltrami dilatation on the annulus Π_j by the formula
$$\left(\frac{\mathbf{i} - \lambda_j}{\mathbf{i} + \lambda_j}\right) \frac{|\varphi|}{\varphi}.$$

- The map $\mathcal{E}$ is a holomorphic embedding and the restriction of $\mathcal{E}$ on the diagonal in $\mathbb{H}^k$ is the Teichmüller disc τ^{φ}.

Teichmüller Polydics

- The map $\mathcal{E}$ is a holomorphic embedding and the restriction of $\mathcal{E}$ on the diagonal in $\mathbb{H}^k$ is the Teichmüller disc τ^{φ}.
- The map $\mathcal{E}$ is not proper!

Teichmüller Polydics

- The map $\mathcal{E}$ is a holomorphic embedding and the restriction of $\mathcal{E}$ on the diagonal in $\mathbb{H}^k$ is the Teichmüller disc τ^{φ}.
- The map $\mathcal{E}$ is not proper!
- When φ is a Jenkins-Strebel differential with the maximal number of cylinders two cylinders ($k = 3g - 3 + n$) then $\mathcal{E}(\mathbb{H}^k)$ is an open subset of $\mathcal{T}_{g,n}$.

The Criterion

Theorem

$\mathcal{T}_{g,n}$ holomorphically retracts onto a Teichmüller disc τ^{φ} if and only if the function $\mathbf{\Phi} : \mathcal{E}(\mathbb{H}^k) \to \mathbb{H}$ given by

$$\mathbf{\Phi}(\tau) = \alpha_1\lambda_1 + \cdots + \alpha_k\lambda_k,$$

where $\tau = \mathcal{E}(\lambda_1, ..., \lambda_k)$, can be extended to a holomorphic function $\mathbf{\Phi} : \mathcal{T}_{g,n} \to \mathbb{H}$. Here

$$\alpha_j = \frac{\int\limits_{\Pi_j} |\varphi|}{\int\limits_{S} |\varphi|}.$$

L-shaped Pillowcases

- We apply the above criterion to Teichmüller discs in $\mathcal{T}_{0,5}$ arising from L-shaped pillowcases.

L-shaped Pillowcases

- We apply the above criterion to Teichmüller discs in $\mathcal{T}_{0,5}$ arising from L-shaped pillowcases.
- For $a > 0$, $b \geq 0$ and $0 < q < 1$ we let $L(a, b, q)$ denote the L-shaped polygon as in figure below.

L-shaped polygon

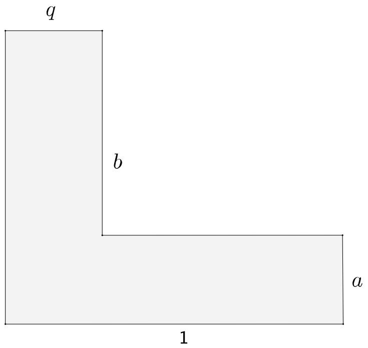

Figure: L-shaped polygon $L(a, b, q)$

The Differential $\psi(a, b, q)$

- The L-shaped pillowcase $S(a, b, q)$ is the double of the polygon $L(a, b, q)$ (it is formally defined as a half-translation surface).

The Differential $\psi(a, b, q)$

- The L-shaped pillowcase $S(a, b, q)$ is the double of the polygon $L(a, b, q)$ (it is formally defined as a half-translation surface).
- The $(2, 0)$ form dz^2 on $L(a, b, q)$ gives rise to the Jenkins-Strebel quadratic differential $\psi(a, b, q)$ on $S(a, b, q)$.

The Differential $\psi(a, b, q)$

- When $b > 0$, the differential $\psi(a, b, q)$ has five first order poles and one zero.

The Differential $\psi(a, b, q)$

- When $b > 0$, the differential $\psi(a, b, q)$ has five first order poles and one zero.
- When $b = 0$ then $\psi(a, b, q)$ has four first order poles and no zeroes.

L-shaped polygon

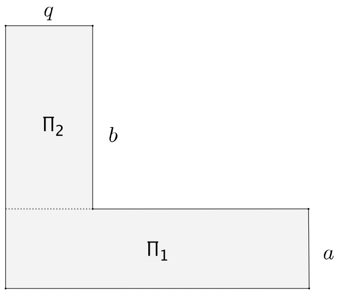

Figure: The differential $\psi(a, b, q)$

The Function Ψ

- $\mathcal{T}_{0,5}$ holomorphically retracts onto the Teichmüller disc generated by a zero free Jenkins-Streel differential on the five punctures sphere ($\psi(a, 0, q)$ is an example of such differential). Every such differential contains a single cylinder.

The Function $\mathbf{\Psi}$

- $\mathcal{T}_{0,5}$ holomorphically retracts onto the Teichmüller disc generated by a zero free Jenkins-Streel differential on the five punctures sphere ($\psi(a,0,q)$ is an example of such differential). Every such differential contains a single cylinder.
- In fact, the forgetful map

$$\mathbf{\Psi} : \mathcal{T}_{0,5} \to \mathcal{T}_{0,4} \equiv \mathbb{H}$$

is the corresponding retraction.

The Function $\boldsymbol{\Psi}$

- Assuming $\mathcal{T}_{0,5}$ retracts onto each Teichmüller disc, for each $q_0 \in (0,1)$ we can find the corresponding holomorphic function (using the above mentioned criterion)

$$\boldsymbol{\Psi} : \mathcal{T}_{0,5} \to \mathbb{H}.$$

The Function $\mathbf{\Psi}$

- Assuming $\mathcal{T}_{0,5}$ retracts onto each Teichmüller disc, for each $q_0 \in (0,1)$ we can find the corresponding holomorphic function (using the above mentioned criterion)

$$\mathbf{\Psi} : \mathcal{T}_{0,5} \to \mathbb{H}.$$

- The restriction of $\mathbf{\Psi}$ to the locus $S(a,b,q_0)$ (we vary a and b but q_0 remains fixed) is the computed

$$\mathbf{\Psi}\big(S(a,b,q_0)\big) = (a + bq_0)\mathbf{i}.$$

$\mathbf{\Psi}$ is not holomorphic

- It can be shown that such a map $\mathbf{\Psi}$ can not be holomorphic at points $S(a, 0, q_0) \in \mathcal{T}_{0,5}$.

$\mathbf{\Psi}$ is not holomorphic

- It can be shown that such a map $\mathbf{\Psi}$ can not be holomorphic at points $S(a, 0, q_0) \in \mathcal{T}_{0,5}$.

- The points $S(a, 0, q_0)$ lie in the boundary of the poly-plane $\mathcal{E}(\mathbb{H}^2)$.

Ψ is not holomorphic

- This is shown by considering the smooth path $S(t) = S(a, 0, q_0 - t)$ in $\mathcal{T}_{0,5}$.

Ψ is not holomorphic

- This is shown by considering the smooth path $S(t) = S(a, 0, q_0 - t)$ in $\mathcal{T}_{0,5}$.
- This path corresponds to moving the fifth marked point on $\mathbb{S}^2$ while keeping the other four points fixed.

The Path $S(t)$

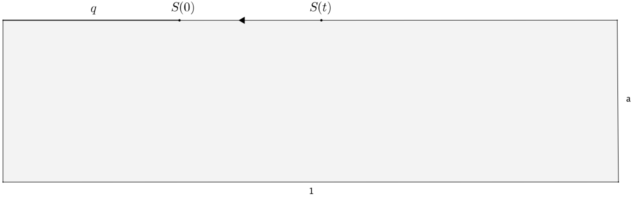

Figure: The Path $S(t)$

$\mathbf{\Psi}$ is not holomorphic

- We compute the first and second derivatives of $\mathbf{\Psi}\big(S(t)\big)$ and verify that $\mathbf{\Psi}$ is not twice differentiable at $S(0)$.

$\boldsymbol{\Psi}$ is not holomorphic

- We compute the first and second derivatives of $\boldsymbol{\Psi}\big(S(t)\big)$ and verify that $\boldsymbol{\Psi}$ is not twice differentiable at $S(0)$.

- Using the explicit formula for $\boldsymbol{\Psi}$ and the Schwarz-Christoffel formula we get the following lemma.

$\boldsymbol{\Psi}$ is not holomorphic

Lemma

$$\boldsymbol{\Psi}\big(S(t)\big) = \boldsymbol{\Psi}\big(S(0)\big) + \beta_1\big(1 + o(1)\big)\frac{t}{\log t^{-1}}$$
$$+ \beta_2\big(1 + o(1)\big)\frac{t^2}{\log t^{-1}} + o\left(\frac{t^2}{\log t^{-1}}\right).$$

$\boldsymbol{\Psi}$ is not holomorphic

Lemma

$$\boldsymbol{\Psi}\big(S(t)\big) = \boldsymbol{\Psi}\big(S(0)\big) + \beta_1\big(1 + o(1)\big)\frac{t}{\log t^{-1}} + \beta_2\big(1 + o(1)\big)\frac{t^2}{\log t^{-1}} + o\left(\frac{t^2}{\log t^{-1}}\right).$$

- Since the path $S(t)$ is infinitely differentiable we verify that $\boldsymbol{\Psi}$ is not twice differentiable at $S(0)$.