

CIRM hybrid conference "Teichmüller Theory: Classical, Higher, Super and Quantum"

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"A double version of Turaev's gate derivatives"

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joint work with

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K : field of char. 0

Σ : compact connected oriented surface with $\partial\Sigma \neq \emptyset$

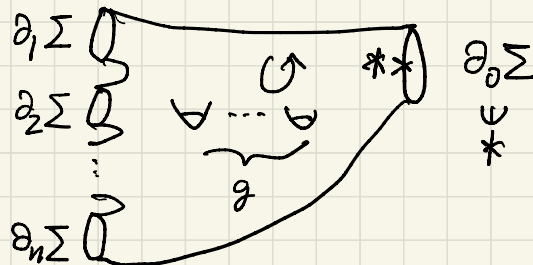
$\implies$
Classification

$\exists g, \exists n \geq 0, \Sigma = \Sigma_{g,n+1} = \partial\Sigma$

$$\chi(\Sigma_{g,n+1}) = 1 - 2g - n$$

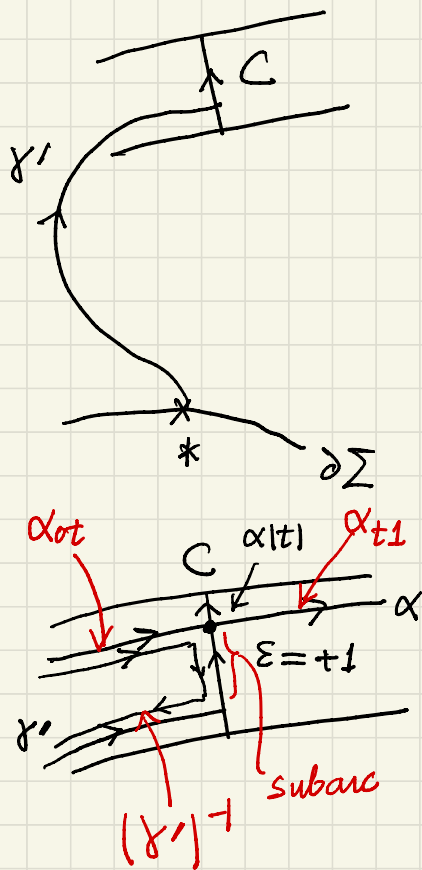
$*$ $\in \partial_0\Sigma$ basepoint

$\pi := \pi_1(\Sigma, *)$ free group of rank $2g+n$



→ more general setting

Turaev (arXiv:1901.02634), gate derivatives



C : gate, an oriented arc in Σ
connecting 2 distinct points on $\partial\Sigma$
away from the basepoint $*$

$\gamma': [0,1], 0,1 \rightarrow (\Sigma, *, C)$ a path

$\partial_C^{\gamma'}: \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi$ gate derivative

$\forall \alpha \in \pi$ Choose its representative transverse to C

$$\partial_C^{\gamma'}(\alpha) \stackrel{\text{def}}{=} \sum_{t \in \alpha^{-1}(C)} \varepsilon_{\alpha|t}(\alpha, C) \alpha_{[t,1]} \left(\begin{array}{c} \text{subarc of } C \\ \text{from } \alpha|t| \\ \text{to } \gamma'(1) \end{array} \right) (\gamma')^{-1} \in \mathbb{Z}\pi$$

local intersection number $\in \{\pm 1\}$

$\alpha_{[0,t]} := \alpha|_{[0,t]}$ segment of α

Fox derivative

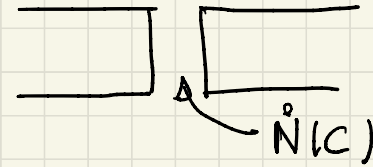
$\forall a, \forall b \in \mathbb{Z}\pi$

$$\partial_C^{\gamma'}(ab) = \partial_C^{\gamma'}(a) \text{aug}(b) + a \partial_C^{\gamma'}(b)$$

$\text{aug}: \mathbb{Z}\pi \rightarrow \mathbb{Z}$
 $\sum \lambda_i \alpha_i \mapsto \sum \lambda_i$
augmentation map

Rmk $\mathring{N}(C)$: open tubular neighborhood

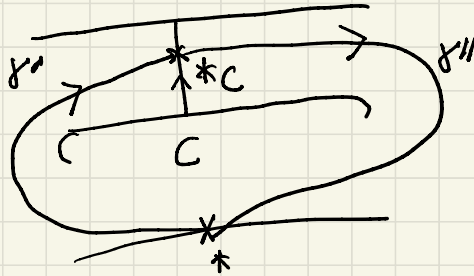
$$\chi(\Sigma \setminus \mathring{N}(C)) = \chi(\Sigma) + 1$$



$$\left. \begin{array}{l} \therefore \{ \Sigma \setminus C, \mathring{N}(C) \} \text{ open covering of } \Sigma, \Sigma \setminus C \simeq \Sigma \setminus \mathring{N}(C) \text{ htpy equiv.} \\ \chi(\Sigma) = \chi(\Sigma \setminus C) + \chi(\mathring{N}(C)) - \chi(\mathring{N}(C) \setminus C) = \chi(\Sigma \setminus \mathring{N}(C)) + 1 - 2 // \end{array} \right\}$$

double version

gate double derivatives



$*_C$: a fixed midpoint of C

$$\gamma' : [0, 1] \rightarrow (\Sigma, *_C, *_C)$$

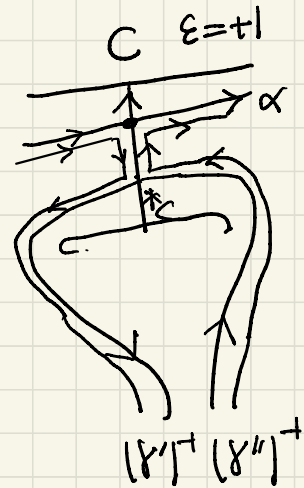
$$\gamma'' : [0, 1] \rightarrow (\Sigma, *_C, *)$$

paths

st. $\gamma'|_{[0, 1]}$ and $\gamma''|_{[0, 1]}$ are disjoint from C

γ', γ'' are transverse to C

$$\varepsilon_{*_C}(\gamma', \gamma'', C) = +1$$



$\alpha \in \Pi$ transverse to C

$$D(\alpha) \stackrel{\text{def}}{=} \sum_{t \in \alpha^{-1}(C)} \varepsilon_{\alpha(t)}(\alpha, C) \alpha_{0t} \left(\begin{array}{l} \text{subarc of } C \\ \text{from } \alpha(t) \\ \text{to } *_C \end{array} \right) (\gamma')^{\dagger} \otimes (\gamma'')^{\dagger} \left(\begin{array}{l} \text{subarc of } C \\ \text{from } *_C \\ \text{to } \alpha(t) \end{array} \right) \alpha_{t1} \in \mathbb{Z} \Pi \otimes \mathbb{Z} \Pi$$

$D : \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi \otimes \mathbb{Z}\pi$ gate double derivative associated to C, γ' and γ''

• double derivation $\forall a, \forall b \in \mathbb{Z}\pi \quad D(ab) = D(a)(1 \otimes b) + (a \otimes 1)D(b)$

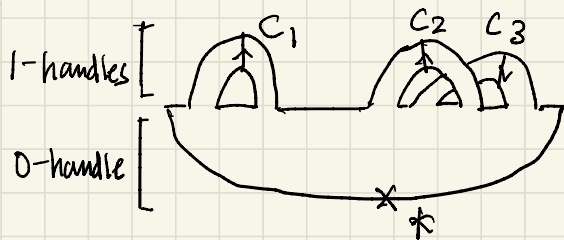
• $\gamma := \gamma' \gamma'' \in \pi, \quad D(\gamma) = \begin{array}{c} \gamma' \quad \gamma'' \\ \text{diagram of two arcs } \gamma' \text{ and } \gamma'' \text{ meeting at a point } * \\ \text{with } \gamma' \text{ on the left and } \gamma'' \text{ on the right} \end{array} = 1 \otimes 1 \in \mathbb{Z}\pi \otimes \mathbb{Z}\pi$

gate system

Definition $\mathcal{C} = \{C_i\}_{i=1}^m$: a gate system on Σ

- $\iff$ $\begin{cases} 0) C_i : \text{a "gate", an oriented arc in } \Sigma \text{ connecting 2 distinct points on } \partial\Sigma \text{ away from } * \\ 1) C_i \cap C_j = \emptyset \text{ if } i \neq j \\ 2) \sum \bigvee_{i=1}^m \overset{\circ}{N}(C_i) \cong D^2 \text{ (2-dim. disk)} \end{cases}$
- $\nearrow m = \chi(D^2) - \chi(\Sigma) = 2g + n$
- $\nwarrow$ open tubular nbd $\rightarrow$ 0-handle

$\longleftrightarrow$ (handle decomposition of Σ with a single 0-handle and no 2-handles)

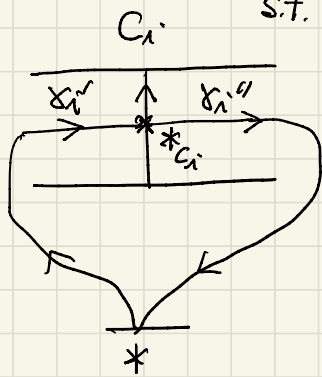


$\mathcal{C} = \{C_i\}_{i=1}^{2g+n}$ a gate system on $\Sigma (\cong \Sigma_{g,n})$ $*_{C_i} \in C_i$ a fixed midpoint

Choose γ_i' and γ_i'' ($1 \leq i \leq 2g+n$) $\gamma_i': ([0,1], 0,1) \rightarrow (\Sigma, *, *_{C_i})$ paths
 $\gamma_i'': ([0,1], 0,1) \rightarrow (\Sigma, *_{C_i}, *)$

s.t. $\gamma_i'([0,1])$ and $\gamma_i''([0,1])$ are disjoint from all C_j 's

$\gamma_i := \gamma_i' \gamma_i''$ is transverse to C_i , $\epsilon_{*_{C_i}}(\gamma_i, C_i) = +1$



- $\{\gamma_i\}_{i=1}^{2g+n}$ unique up to homotopy ($\because D^2 \simeq *$)

- $\{\gamma_i\}_{i=1}^{2g+n} \subset \pi = \pi_1(\Sigma, *)$ free generator system

- $\frac{\partial}{\partial C_i} := \begin{pmatrix} \text{double gate derivative} \\ \text{associated with } C_i, \gamma_i' \text{ and } \gamma_i'' \end{pmatrix} : \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi \otimes \mathbb{Z}\pi$
 double derivations

$$\frac{\partial}{\partial C_i}(\gamma_j) = \delta_{ij} (1 \otimes 1) \quad (1 \leq i, j \leq 2g+n)$$

Notation : trace space (Lie algebra abelianization), K : field of char. 0.

A : (topological) associative K -algebra $\ni 1$

$\leadsto A$: Lie algebra with $[a, b] := ab - ba$ ($a, b \in A$)

$|A| := A/[A, A]$: "trace space" (Lie algebra abelianization) K -vec. sp.

where $[A, A] :=$ (the closure of) the K -linear span of $\{[a, b] : a, b \in A\}$

$| \cdot | : A \rightarrow |A|$, $a \mapsto |a|$, the quotient map, $1 := |1| \in |A|$ $\pi_1(\Sigma, *)$

ex. 1) $K\pi = \left\{ \sum_{\alpha \in \pi} \lambda_\alpha \alpha : \lambda_\alpha \in K, \lambda_\alpha = 0 \text{ except finite } \alpha\text{'s} \right\}$ group ring of π

$|K\pi| = K(\pi/\text{conj}) = K[S^1, \Sigma]$, $\alpha \in \pi \xrightarrow{| \cdot |} |\alpha| \in [S^1, \Sigma]$ free homotopy class

similarly $|\mathbb{Z}\pi| = \mathbb{Z}[S^1, \Sigma]$ $\star$ free homotopy set of free loops on Σ

ex. 2) V : fin. dim. K -vector space

$\hat{T}(V) := \prod_{m=0}^{\infty} V^{\otimes m}$ completed tensor algebra $\left\{ \begin{array}{l} \text{complete Hopf algebra} \\ \text{topology on } \hat{T}(V) \\ \Leftarrow \text{degree filtration} \end{array} \right.$

$|\hat{T}(V)| = \prod_{m=0}^{\infty} (V^{\otimes m})_{\mathbb{Z}/m}$ cyclic coinvariants | space of "cyclic words" in V

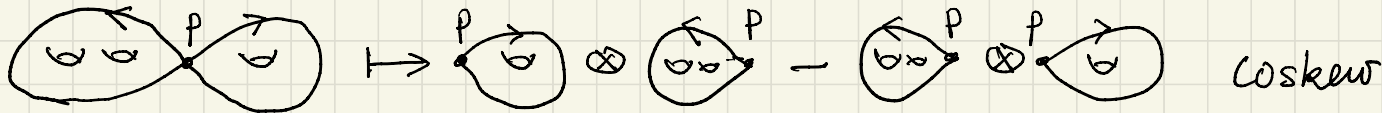
Turaev cobracket and its framed version

$\alpha \in \pi/\text{conj} = [S^1, \Sigma]$ free loop on Σ , represented by a generic immersion

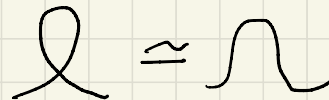
$$D_\alpha := \{ (t_1, t_2) \in S^1 \times S^1 : t_1 \neq t_2, \alpha|_{t_1} = \alpha|_{t_2} \}$$

$$\delta \alpha := \sum_{(t_1, t_2) \in D_\alpha} \varepsilon(\dot{\alpha}|_{t_1}, \dot{\alpha}|_{t_2}) |\alpha|_{t_1 t_2} \otimes |\alpha|_{t_2 t_1} \in \mathbb{Z}\pi \otimes \mathbb{Z}\pi \quad \text{Turaev cobracket}$$

(where $\alpha|_{t_1 t_2} := \alpha|_{[t_1, t_2]}$, $\alpha|_{t_2 t_1} := \alpha|_{[t_2, t_1]}$, $[t'', t'] = [t'', 1] \cup [t', 1]$ if $t' < t''$)



not well-defined $\Leftarrow$ monogon



original definition:

take the quotient $|\mathbb{Z}\pi|' := |\mathbb{Z}\pi| / \mathbb{Z}1$

$(|\mathbb{Z}\pi|', \text{Goldman bracket}, \delta) : \text{Lie bialgebra (Turaev)}$

involutive (Chas) $\left(\begin{smallmatrix} \text{Goldman} \\ \text{bracket} \end{smallmatrix} \circ \delta = 0 \right)$

framed version fix a framing $T\Sigma \cong \Sigma \times \mathbb{R}^2$ ($\because \partial\Sigma \neq \emptyset$)

$\leadsto$ rotation number of immersed loops rot_f

$$\delta^f : |\mathbb{Z}\pi| \rightarrow |\mathbb{Z}\pi| \otimes |\mathbb{Z}\pi| \quad (\text{rot}_f(Q) = \text{rot}_f(\int \setminus) + 1)$$

$$\delta^f \alpha := \delta \alpha + (\text{rot}_f \alpha) (1 \otimes \alpha - \alpha \otimes 1) \in |\mathbb{Z}\pi| \otimes |\mathbb{Z}\pi|$$

well-defined

$\leadsto$ involutive Lie bialgebra

formal description of δ^f

$$H := H_1(\Sigma; \mathbb{K}) = H^{(1)} \supset H^{(2)} := \text{Image}(\text{ind}_* : H_1(\partial\Sigma; \mathbb{K}) \rightarrow H_1(\Sigma; \mathbb{K}))$$

$$\text{gr } H = (H^{(1)} / H^{(2)}) \oplus H^{(2)}$$

$\exists \{F_p\}_{p=0}^\infty$ decreasing filtration on $\mathbb{K}\pi$

$$\text{s.t. } \text{gr}(\mathbb{K}\pi) := \prod_{p=0}^\infty (F_p / F_{p+1}) \stackrel{\text{canon}}{\cong} \widehat{T}(\text{gr } H)$$

$\nwarrow$ completed graded quotient

Theorem (Alekseev-K.-Kuno-Naef) $\text{genus}(\Sigma) \neq 1$ or (genus 1 except some f)
 $\Rightarrow \exists \theta: K\pi \rightarrow \hat{T}(\text{gr}H)$ filtration preserving Hopf algebra homomorphism
 s.t. $\text{gr} \theta = 1_{\hat{T}(\text{gr}H)}$ ($\leadsto \theta: \hat{K}\pi \xrightarrow{\sim} \hat{T}(\text{gr}H)$ isom) completion w.r.t to $\{F_p\}$
 $|\theta|: (|\hat{K}\pi|, \text{Goldman bracket}, \delta^f) \xrightarrow{\sim} (\hat{T}(\text{gr}H), \text{gr}(\text{Goldman bracket}), \text{gr}(\delta^f))$
 Lie bialgebra isom.

Rmk 1 - genus 0, Massuyeau using the Kontsevich integral before AKKN

- Alekseev-Naef KZ connection ($g=0$)

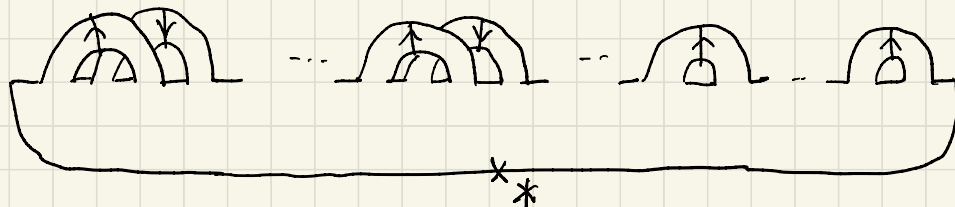
- Hain Mixed Hodge Structure ($\forall g \geq 0, g=1$ except some f)

Rmk 2 θ is a solution to the Kashiwara-Vergne problem (consisting of 2 cond's (KVI) and (KVII))
 Goldman bracket $\leftrightarrow$ (KVI)
 Turaev cobracket $\leftrightarrow$ (KVI) + (KVII)
 $\uparrow$
special/symplectic condition on θ

Rmk 3 (K.-Kuno, Massuyeau-Turaev θ : (conjugate of) special/symplectic
 $\Rightarrow |\theta|$: Lie algebra isom
Alekseev-K.-Kuno-Naef ($\Leftarrow$)

Key formula for the formal description of δ^f

$$A := \hat{T}(\text{gr} H)$$



← "standard" gate system

$\Rightarrow \{\alpha_i, \beta_i, \gamma_j\} \subset \pi$ standard free generator system

$\exists! f_0: T\Sigma \xrightarrow{\cong} \Sigma \times \mathbb{R}^2$ framing

$$\text{s.t. } \begin{cases} \text{rot}_{f_0} |\alpha_i| = \text{rot}_{f_0} |\beta_i| = 0 \\ \text{rot}_{f_0} |\gamma_j| = 0 \end{cases} \quad \begin{matrix} \text{original} \\ (= -1) \end{matrix}$$

$$x_i := [\alpha_i] \bmod H^{(2)}, \quad y_i := [\beta_i] \bmod H^{(2)}, \quad z_j := [\gamma_j]$$

$$\theta_{\exp}: \mathbb{K}\pi \rightarrow A = \hat{T}(\text{gr} H), \quad \alpha_i \mapsto e^{x_i}, \quad \beta_i \mapsto e^{y_i}, \quad \gamma_j \mapsto e^{z_j}$$

$$(\Rightarrow |\theta_{\exp}|: |\mathbb{K}\pi| \xrightarrow{\cong} |A| \text{ K-rep sp, sum})$$

$$g\text{Div}^{f_0} \doteq \text{Aleksiev-Torossian divergence} \leftrightarrow [KVII]$$

$$\sigma_{\exp} \leftrightarrow [KVI]$$

$$\boxed{\text{AKKN (original proof: algebraic)}} \\ \delta_{\exp}^{f_0} = g\text{Div}^{f_0} \circ \sigma_{\exp}$$

$$\text{where } \delta_{\exp}^{f_0}: (|\theta_{\exp}| \otimes |\theta_{\exp}|) \circ \delta_{f_0}^{f_0} |\theta_{\exp}|^{-1}: |A| \rightarrow |A| \otimes |A|$$

$$\sigma_{\exp}: (\theta_{\exp})_* \circ \sigma \circ |\theta_{\exp}|^{-1}: |A| \rightarrow \text{Der} |A|$$

where σ : based version of the Goldman Bracket (K.-Kimura)

(Rmk If $\text{rot}_f |\gamma_j| \neq 0$ for some $j \geq 1$, (will be explained later)
then we need Massuyeau - Turaev's description of the homotopy intersect. form

$$g\text{Div}^{f_0}(u) = \sum_{\substack{w=x_i, y_i, z_j \\ 1 \leq i \leq g \\ 1 \leq j \leq n}} \left| \frac{\partial}{\partial e^w} u|e^w\rangle - 1 \otimes u|e^w\rangle e^{-w} \right| \in |A| \otimes |A|. \quad (u \in \text{Der}(A))$$

where $\frac{\partial}{\partial e^w} : A \rightarrow A \otimes A$ double derivation s.t. $\frac{\partial}{\partial e^w} |e^{w'}\rangle = \delta_{ww'} (1 \otimes 1)$
 $\forall w, w' \in \{x_i, y_i, z_j\}$

$g\text{Div}^{f_0}$ was introduced by an algebraic formula in the context of Lie theory
 $\nwarrow$ topological interpretation? or non-comm. Poisson geometry

$\uparrow$
gate double derivatives $\mathcal{C} = \{C_i\}_{i=1}^{2g+m}$ gate system

$\{\gamma_i\}_{i=1}^{2g+2} \subset \pi$ free generator associated with $\mathcal{C}$

$$\text{Div}^{\mathcal{C}} : \text{Der}(|\mathbb{Z}\pi|) \rightarrow |\mathbb{Z}\pi| \otimes |\mathbb{Z}\pi|$$

$$\text{Div}^{\mathcal{C}}(u) := \left| \sum_{i=1}^{2g+m} \frac{\partial}{\partial C_i} u|\gamma_i\rangle - 1 \otimes u|\gamma_i\rangle \gamma_i^{-1} \right| \in |\mathbb{Z}\pi| \otimes |\mathbb{Z}\pi| \quad (u \in \text{Der}(\mathbb{Z}\pi))$$

$$(\theta_{\exp_*}(\text{Div}^{\mathcal{C}}) = g\text{Div}^{f_0} \text{ if } \mathcal{C} \text{ is "standard"})$$

"Theorem" $\text{Div}^{\mathcal{C}}$ is independent of the choice of $\mathcal{C}$ up to framing terms.

"proof" Compute the effects of the following 2 "coordinate changes"

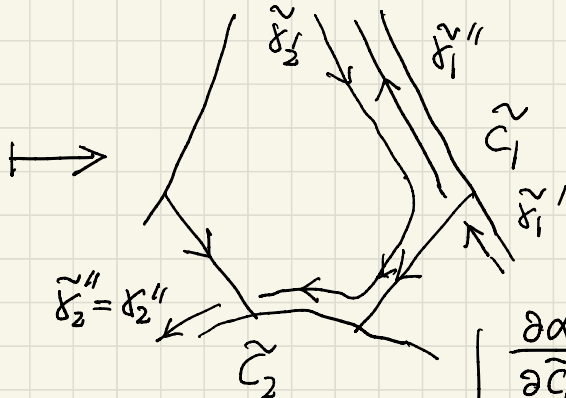
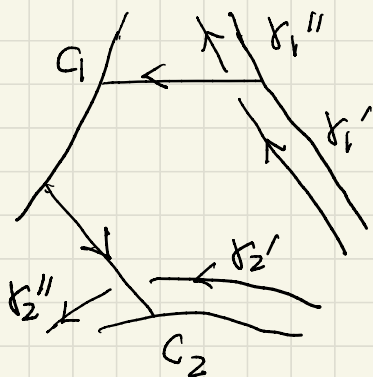
① reversing the orientation of a gate

$$\mathcal{C} \quad \begin{array}{|c|} \hline \xrightarrow{x'} \quad \uparrow \quad \xrightarrow{x''} \\ \hline \end{array} \quad \mapsto \quad \overline{\mathcal{C}} \quad \begin{array}{|c|} \hline \xleftarrow{x'} \quad \downarrow \quad \xleftarrow{x''} \\ \hline \end{array} \quad \begin{array}{l} \overline{x'} := (x')^\top \\ \overline{x''} := (x'')^\top \\ \overline{C} := C^{-1} \end{array} \quad , \quad x \mapsto \overline{x} = x^\top$$

$$\text{Then } \frac{\partial \alpha}{\partial \overline{C}} = - (1 \otimes x) \frac{\partial \alpha}{\partial C} (x \otimes 1) \quad (\forall \alpha \in \pi)$$

$$\begin{aligned} (\text{Div}^{\overline{\mathcal{C}}} - \text{Div}^{\mathcal{C}})|u| &= \left| \frac{\partial u|x^{-1}|}{\partial \overline{C}} - 1 \otimes u|x^\top| x - \frac{\partial u|x|}{\partial C} + 1 \otimes u|x| x^\top \right| \\ &= \left| -1 \otimes u|x| x^\top - x^\top u|x| \otimes 1 - 1 \otimes u|x^\top| x + 1 \otimes u|x| x^\top \right| \\ &= 1 \otimes |u|x| x^\top| - |u|x| x^\top| \otimes 1 \\ &= u(1 \wedge |\log x|) \quad \leftarrow \text{--- framing term.} \end{aligned}$$

② elementary move $\mathcal{C} \mapsto \tilde{\mathcal{C}}, [C_1, C_2] \mapsto [\tilde{C}_1, \tilde{C}_2]$



$$\tilde{t}_2' = t_1' t_2'$$

$$\begin{cases} \tilde{t}_1' = t_1 \\ \tilde{t}_2' = t_1' t_2 \end{cases}$$

$$(\text{Div}^{\tilde{\mathcal{C}}} - \text{Div}^{\mathcal{C}})(u) = u(1 \wedge |\log \delta_1|)$$

Δ
framing term.

$$\begin{cases} \frac{\partial \alpha}{\partial \tilde{C}_1} = \frac{\partial \alpha}{\partial C_1} + (1 \otimes t_1' t_2) \frac{\partial \alpha}{\partial C_2} \\ \frac{\partial \alpha}{\partial \tilde{C}_2} = \frac{\partial \alpha}{\partial C_2} (t_2 \otimes 1) \end{cases}$$

Rmk If we change a framing f for $g \text{Div}^f$ fixing the restriction $f|_{\partial \Sigma}$,

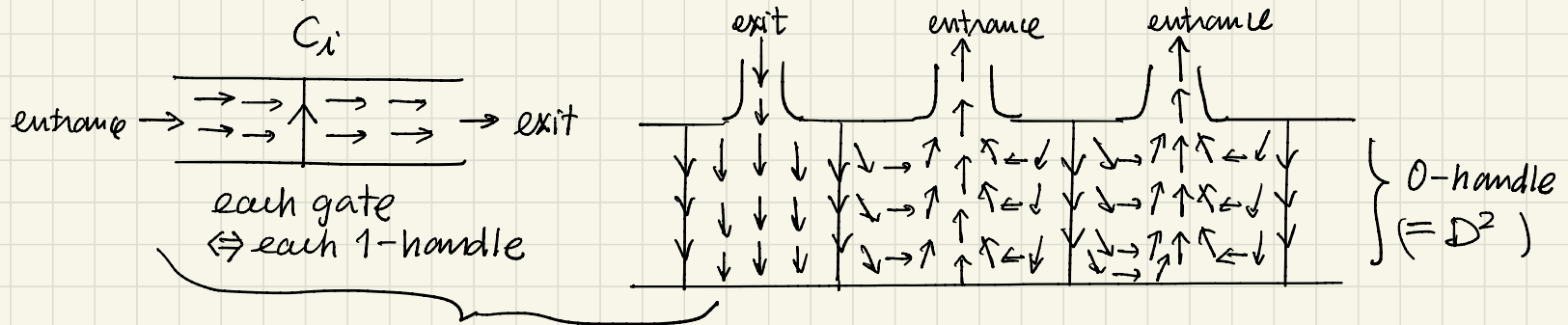
then the difference is given by $u \in \text{Der}(A) \mapsto u(1 \wedge w)$ for some $w \in \text{gr}H$

For any $\gamma \in \pi$, $|\log \gamma|$ can be identified with its homology class.

Thus we call $u \mapsto u(1 \wedge |\log \gamma|)$ a "framing term".

framing associated with a gate system $\mathcal{C} = \{C_i\}_{i=1}^{2g+n}$

framing: $T\Sigma \xrightarrow{\cong} \Sigma \times \mathbb{R}^2$ ori-pres isomorphism $\Leftarrow$ nowhere zero vector field on Σ



$f_{\mathcal{C}}$: framing associated with $\mathcal{C} \Rightarrow \text{rot}_{\mathcal{C}} := \text{rot}_{f_{\mathcal{C}}}$ rotation number associated with $f_{\mathcal{C}}$

(Rmk 1) $\text{rot}_{\mathcal{C}}(\partial_j \Sigma) = 0 \quad (1 \leq j \leq n)$ (Poincaré-Hopf $\text{rot}_{\mathcal{C}}(\partial \Sigma) = \chi(\Sigma) = 1 - 2g - n$)

2) If $\mathcal{C}$ is "standard", then $f_{\mathcal{C}} = f_0$ (i.e, $\text{rot}_{\mathcal{C}}(K_i) = \text{rot}_{\mathcal{C}}(B_i) = 0$)

Theorem [AKKN: topological proof of our previous result]

$\forall \mathcal{C}$: gate system.

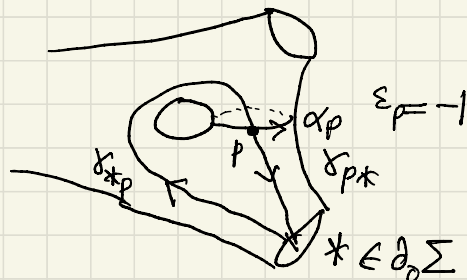
$$\delta^{f_{\mathcal{C}}} = \text{Div}^{\mathcal{C}} \circ \sigma : |\mathbb{Z}\Pi| \rightarrow |\mathbb{Z}\Pi| \otimes |\mathbb{Z}\Pi|$$

Here $\sigma: |\mathbb{Z}\pi| \rightarrow \text{Der}(\mathbb{Z}\pi)$ based version of the Goldman bracket (K.-Kuno).

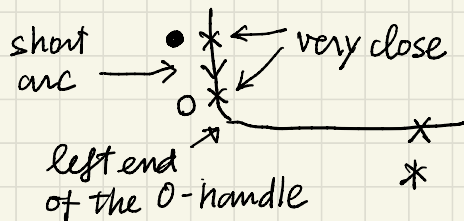
$\alpha \in [S^1, \Sigma]$, $\gamma \in \pi$, in general position

$$\sigma(\alpha)(\gamma) = \sum_{p \in \alpha \cap \gamma} \varepsilon_p(\alpha, \gamma) \gamma_{*p} \alpha_p \gamma_{p*} \in \mathbb{Z}\pi$$

$\varepsilon_p(\alpha, \gamma) \in \{\pm 1\}$ local intersection no. at p



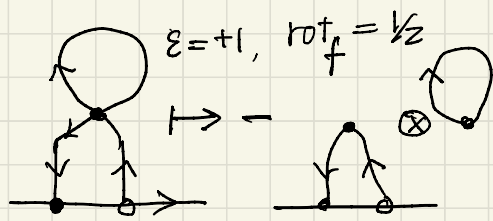
Proof of Theorem $\Leftarrow$ "dual" normal form of a based loop $\alpha \in \pi_1(\Sigma, *)$, α^+ and α^-



Take 2 basepoints $\bullet$ and $o \in \partial_0 \Sigma$ other than $*$

$\Rightarrow$ One can identify all of $\pi_1(\Sigma, \bullet)$, $\pi_1(\Sigma, o)$, $\pi_1(\Sigma, \bullet, o) = \pi_1(\Sigma(\bullet, o))$ and $\pi_1(\Sigma, o, \bullet)$ using this short arc.

2 based versions of the Turaev cobracket $\mu_{\bullet o}^f$ and $\mu_{o \bullet}^f$



$\gamma: ([0,1], 0, 1) \rightarrow (\Sigma, o, \bullet)$ generic immersion

$$\mu_{o \bullet}^f(\gamma) := - \sum_{\substack{t_1 < t_2 \\ \gamma(t_1) = \gamma(t_2)}} \varepsilon(\dot{\gamma}(t_1), \dot{\gamma}(t_2)) \gamma_{o t_1} \gamma_{t_2 \bullet} \otimes |\gamma_{t_1 t_2}| - (\text{rot}_\gamma f - \frac{1}{2}) \gamma \otimes 1$$

$$\in \mathbb{Z}\pi_1(\Sigma, o, \bullet) \otimes |\mathbb{Z}\pi|$$

$\gamma: ([0,1], 0, 1) \rightarrow (\Sigma, \bullet, o)$ generic immersion

$$\mu_{\bullet o}^f(\gamma) := + \sum_{\substack{t_1 < t_2 \\ \gamma(t_1) = \gamma(t_2)}} \varepsilon(\dot{\gamma}(t_1), \dot{\gamma}(t_2)) |\gamma_{t_1 t_2}| \otimes \gamma_{\bullet t_1} \gamma_{t_2 o} + (\text{rot}_\gamma f + \frac{1}{2}) 1 \otimes \gamma$$

$$\in |\mathbb{Z}\pi| \otimes \mathbb{Z}\pi_1(\Sigma, \bullet, o)$$

$\text{rot}_f \gamma = -\frac{1}{2}, \varepsilon = -1$

$$\text{If } \pi_1(\Sigma, o, \bullet) \cong \pi_1(\Sigma, \bullet, o) \cong \pi_1(\Sigma, \bullet, \bullet),$$

$$\Downarrow \alpha^+ \mapsto \alpha \longleftarrow \alpha^- \Downarrow$$

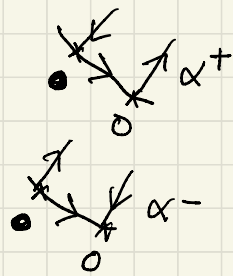
then

$$\delta^f |\alpha| = (1 \otimes 1) \mu_{o \bullet}(\alpha^+) + (1 \otimes 1) \mu_{\bullet o}(\alpha^-) \in |\mathbb{Z}\pi| \otimes |\mathbb{Z}\pi|$$

Normal forms

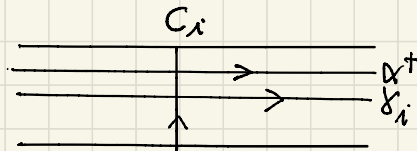
$$\alpha \in \pi_1(\Sigma, \bullet) \mapsto \begin{cases} \text{normal form } \alpha^+ \in \pi_1(\Sigma, o, \bullet) \\ \text{dual normal form } \alpha^- \in \pi_1(\Sigma, \bullet, o) \end{cases}$$

s.t. $\alpha \simeq \alpha^+ \simeq \alpha^-$



normal form $\alpha^+ : ([0, 1], o, 1) \rightarrow (\Sigma, o, \bullet)$ piecewise C^∞ immersion

1) Whenever α^+ intersects C_i , it crosses all the 1-handle dual to C_i



2) 3)
There is a surveillance camera on each gate

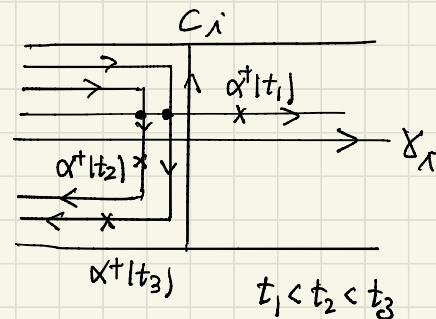
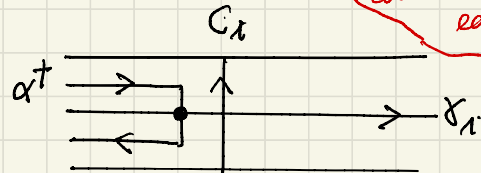
2) α^+ may intersect γ_i only near the gate C_i

Then it has to turn around in front of the gate C_i

3) α^+ may intersect itself only near some gate C_i

But may not any segments in 2)

Then the "junior" segment is parallel to the gate and "outer" than all the "senior" ones



outside
↑↑

$$t_1 < t_2 < t_3$$

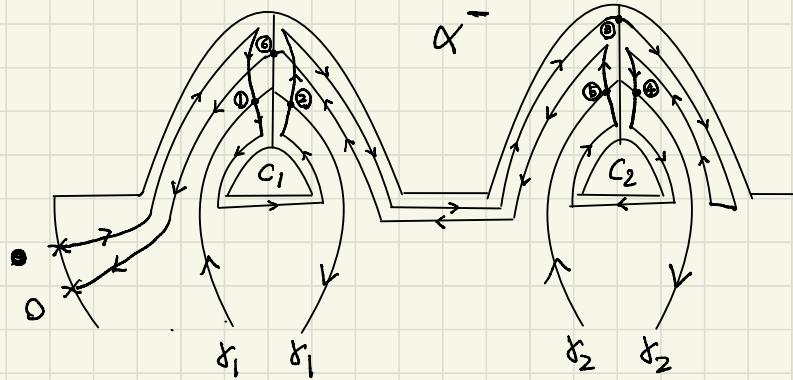
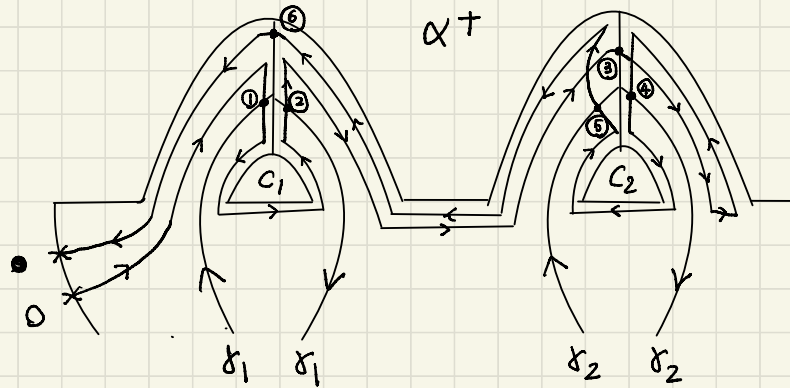
dual normal form $\alpha^- : ([0, 1], o, 1) \rightarrow (\Sigma, \bullet, o)$ $\alpha^-(t) := (\alpha^+)^+(1-t), (0 \leq t \leq 1)$
piecewise C^∞ immersion

example

normal form

$$\alpha = \delta_2 \delta_1^{-1}$$

dual normal form



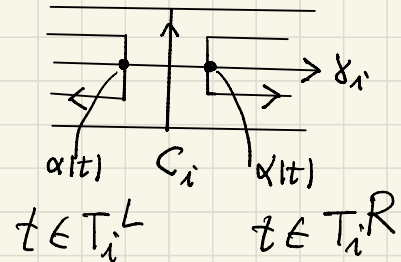
Observations

1) Can choose the parameters of α^+ and α^-

such that the times α^+ intersects C_i and δ_i are the same as α^-

$$S_i := \{s \in]0,1[: \alpha^\pm(s) \in C_i\}, \quad T_i := \{t \in]0,1[: \alpha^\pm(t) \in \delta_i\} = T_i^R \cup T_i^L$$

$$\left(\begin{array}{l} \text{ex) } S_1 = \{\textcircled{6}\}, S_2 = \{\textcircled{3}\}, \\ T_1^L = \{\textcircled{1}\}, T_1^R = \{\textcircled{2}\}, T_2^L = \{\textcircled{5}\}, T_2^R = \{\textcircled{4}\} \end{array} \right)$$



$$2) \text{ Set } D_i^+ := \{(s, t) \in S_i \times T_i : s < t\}$$

$$D_i^- := \{(s, t) \in S_i \times T_i : s > t\}$$

Then

$$\bigcup_{i=1}^{2g+n} D_i^+ \cong \{\text{self-intersection of } \alpha^+\}$$

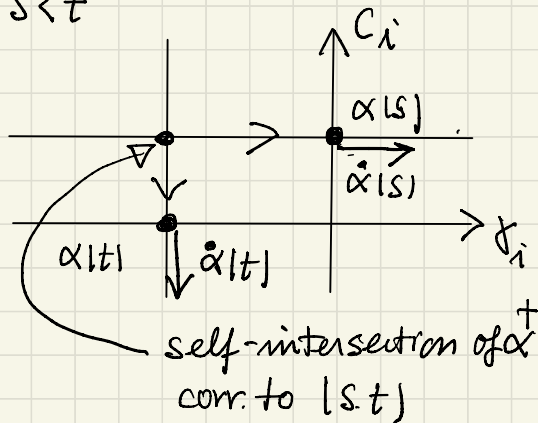
$$\bigcup_{i=1}^{2g+n} D_i^- \cong \{\text{self-intersection of } \alpha^-\}$$

$$\nabla (s, t) \in S_i \times T_i$$

$$\varepsilon_{\alpha(s)}(\alpha, x_i) \varepsilon_{\alpha(t)}(\alpha, C_i) = -\varepsilon(\dot{\alpha}(s), \dot{\alpha}(t))$$

$$(s, t) \in D_i^+ \subset S_i \times T_i$$

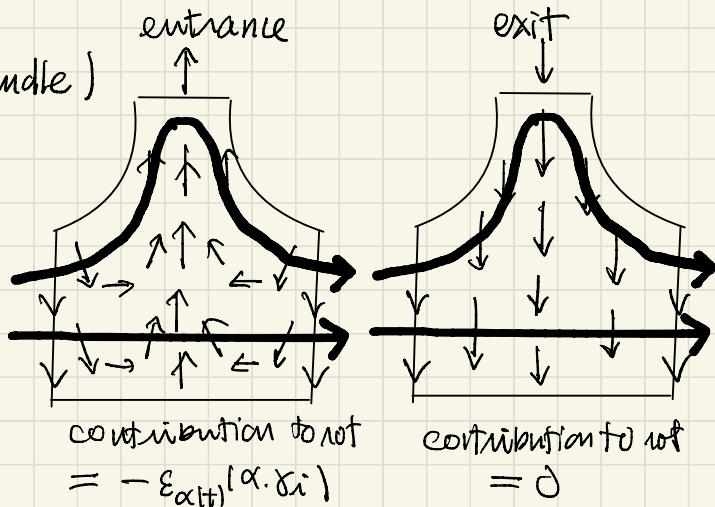
$$s < t$$



$$3) \text{ rot}_C = \text{rot}_{f_C} (\Leftarrow \text{entrance of each 1-handle})$$

$$\text{rot}_C \alpha^+ - \frac{1}{2} = - \sum_{i=1}^{2g+n} \sum_{t \in T_i^L} \varepsilon_{\alpha(t)}(\alpha^+, x_i)$$

$$\text{rot}_C \alpha^- + \frac{1}{2} = - \sum_{i=1}^{2g+n} \sum_{t \in T_i^L} \varepsilon_{\alpha(t)}(\alpha^-, x_i)$$



$$\forall \alpha \in \pi_1(\Sigma, \cdot), |\alpha| \in |\mathbb{Z}\pi|$$

Computation of $\delta^{fe}(|\alpha|) = (1| \otimes 1) \mu_{o \bullet}^{fe}(\alpha^+) + (1 \otimes 1|) \mu_{\bullet o}^{fe}(\alpha^-) \quad (\Leftarrow 2) + 3)$

$$\mu_{o \bullet}^{fe}(\alpha^+) = - \sum_{i=1}^{2g+n} \sum_{1 \leq t_1 \in D_i^+} \varepsilon(\dot{\alpha}(s), \dot{\alpha}(t_1)) \alpha_{o_s} \alpha_{t_1} \otimes |\alpha_{s+}| + \sum_{i=1}^{2g+n} \sum_{t \in T_i^L} \varepsilon_{\alpha(t)}(\alpha, x_i) \alpha \otimes 1$$

$$\mu_{\bullet o}^{fe}(\alpha^-) = + \sum_{i=1}^{2g+n} \sum_{1 \leq t_1 \in D_i^-} \varepsilon(\dot{\alpha}(s), \dot{\alpha}(t_1)) |\alpha_{t_s}| \otimes \alpha_{ot} \alpha_{s1} - \sum_{i=1}^{2g+n} \sum_{t \in T_i^L} \varepsilon_{\alpha(t)}(\alpha, x_i) 1 \otimes \alpha$$

Computation of $\text{Div}^E \sigma(|\alpha|)$

$$\sigma(|\alpha|)(x_i) = \sum_{t \in T_i} \varepsilon_{\alpha(t)}(\alpha, x_i) (x_i)_{* \alpha(t)} \alpha_{\alpha(t)}(x_i)_{\alpha(t)*}$$

$$\left| \frac{\partial}{\partial c_i} (\sigma(|\alpha|)(x_i)) \right| - 1 \otimes |\sigma(|\alpha|)(x_i) x_i^+|$$

$$= \left| \sum_{t \in T_i} \varepsilon_{\alpha(t)}(\alpha, x_i) (x_i)_{* \alpha(t)} \left(\frac{\partial \alpha_{\alpha(t)}}{\partial c_i} \right) (x_i)_{\alpha(t)*} \right|$$

$$+ \sum_{t \in T_i^L} \varepsilon_{\alpha(t)}(\alpha, x_i) |\alpha| \otimes 1 + \sum_{t \in T_i^R} \varepsilon_{\alpha(t)}(\alpha, x_i) 1 \otimes |\alpha| - \sum_{t \in T_i} \varepsilon_{\alpha(t)}(\alpha, x_i) 1 \otimes |\alpha|$$

$$= \left| \sum_{t \in T_i} \varepsilon_{\alpha(t)}(\alpha, x_i) (x_i)_{* \alpha(t)} \left(\frac{\partial \alpha_{\alpha(t)}}{\partial c_i} \right) (x_i)_{\alpha(t)*} \right| + \sum_{t \in T_i^L} \varepsilon_{\alpha(t)}(\alpha, x_i) (|\alpha| \otimes 1 - 1 \otimes |\alpha|)$$

(with $1| \otimes 1$
or $1 \otimes 1|$)
equal

$$\begin{aligned}
& \Downarrow 2) \\
& \sum_{t \in T_i} \varepsilon_{\alpha(t)} |\alpha, x_i| |x_i|_{* \alpha(t)} \left(\frac{\partial \alpha_{\alpha(t)}}{\partial c_i} \right) |x_i|_{\alpha(t)*} \\
& = - \sum_{\substack{(s,t) \in D_i^+ \\ s < t}} \varepsilon(\tilde{\alpha}(s), \tilde{\alpha}(t)) x_i' \alpha_{t1} \alpha_{0s} (x_i')^\dagger \otimes (x_i'')^\dagger \alpha_{st} x_i'' \quad \leftarrow \text{main part of } \mu_{\bullet\bullet}^{fe}(\alpha^+) \\
& \quad + \sum_{\substack{(s,t) \in D_i^- \\ s > t}} \varepsilon(\tilde{\alpha}(t), \tilde{\alpha}(s)) x_i' \alpha_{ts} (x_i')^\dagger \otimes (x_i'')^\dagger \alpha_{s1} \alpha_{0t} x_i'' \quad \leftarrow \text{main part of } \mu_{\bullet\bullet}^{fe}(\alpha^-)
\end{aligned}$$

Consequently we obtain $\forall \alpha \in \pi_1(\Sigma, \circ)$

$$\text{Div}^e \sigma(|\alpha|) = (1| \otimes 1) \mu_{\bullet\bullet}^{fe}(\alpha^+) + (1 \otimes 1|) \mu_{\bullet\bullet}^{fe}(\alpha^-) = \delta^{fe}(|\alpha|) \quad //_{\text{Tmm}}$$

By a similar topological method, we can prove

$$\{ \beta, \alpha \} = \sum_{i=1}^{2g+m} \frac{\partial \alpha}{\partial c_i} \diamond \{ \beta, x_i \} \quad (\forall \alpha, \forall \beta \in \pi)$$

Here $\{, \} : \mathbb{Z}\pi \otimes \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi \otimes \mathbb{Z}\pi$ is the homotopy intersection form due to Turaev and Papakyniakopoulos, and

$$(a' \otimes a'') \diamond (b' \otimes b'') := a'b' \otimes b''a'' \quad (a, a', b, b'' \in \mathbb{Z}\pi) \quad //$$