Banach-Mazur stability of von Neumann algebras

Jean ROYDOR (IMJ-PRG, Sorbonne Université)

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C^*-algebras and von Neumann algebras

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Def. :

· A subalgebra $\mathcal{A} \subset \mathbb{B}(H)$ is a C^*-algebra if it is selfadjoint (*i.e.* stable by involution) and norm closed.

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- G acting on $\mathcal{M}$, the crossed product $\mathcal{M} \rtimes_\alpha G$ is vN alg. generated by a copy of $\mathcal{M}$ and unitaries $(u_g)_{g\in G}$ s.t. $u_g u_h = u_{gh}$ and $u_g^* x u_g = \alpha_g(x)$, $x \in \mathcal{M}$.

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- $L(\mathbb{F}_n)$ is isomorphic (but not cb-isom.) to $\mathbb{B}(\ell_2)$. (Pisier'20).

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1. Functional analytic (stability of isometries of a class of Banach spaces),
2. Question on the vanishing of Hochschild cohomology for von Neumann algebras,
3. Kadison-Kastler conjecture.

The commutative case

Th. (Banach-Stone'37) :

Let $T : C(K_1) \to C(K_2)$ surjective linear isometry. Then
$\exists \ \ u \in C(K_2, \mathbb{T})$ and $\tau : K_2 \to K_1$ homeo. s.t. $T(f) = u.(f \circ \tau)$.
In particular, $C(K_1) = C(K_2)$ $*$-isomorphically.

Proof: identify $k \in K$ with δ_k extreme points of $Ball(C(K)^*)$.

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- The answer to the general question in the commutative case is:

Th. (Amir-Cambern'66) :

If $d(C(K_1), C(K_2)) < 2$, then $C(K_1) = C(K_2)$ $*$-isomorphically.

Rmk 1: the constant 2 above is optimal.
Rmk 2: the key point is a geometric property: If $k \neq k' \in K$, then $\|\delta_k - \delta_{k'}\|_{C(K)^*} = 2$.

Noncommutative Banach-Stone

Th. (Kadison '51) :

Let $T : \mathcal{A} \to \mathcal{B}$ surjective linear isometry between two unital C^*-alg. Then $\exists\ u \in \mathcal{B}$ unitary and $J : \mathcal{A} \to \mathcal{B}$ unital $*$-Jordan isomorphism s.t. $T(x) = uJ(x), \quad \forall x \in \mathcal{A}$.

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Let $J : \mathcal{M} \to \mathcal{N}$ unital $*$-Jordan isomorphism between two vN alg. Then $\exists\ p,\ q$ central projections in $\mathcal{M}, \mathcal{N}$ resp., $\exists \pi : p\mathcal{M}p \to q\mathcal{N}q$ $*$-isomorphism and $\exists \rho : (1-p)\mathcal{M}(1-p) \to (1-q)\mathcal{N}(1-q)$ anti-$*$-isomorphism s.t. $J = \pi \oplus \rho$.

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Rmk: At best, two close vN alg. are $*$-Jordan isomorphic.

Main results

Th. 1 (J. R. '20) :

$\mathcal{M}$, $\mathcal{N}$ two von Neumann algebras s.t. $d(\mathcal{M},\mathcal{N}) \leq 1+\epsilon$ with $\epsilon \leq 10^{-46}$. Then for $i = I, II_1, II_\infty, III$, $d(\mathcal{M}_i,\mathcal{N}_i) \leq 1 + 3577\epsilon^{1/4}$.

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Th. 2 (J. R. '20) :

$\mathcal{M}$ von Neumann algebra with predual $L_1(\mathcal{M})$. Suppose that $H^2(\mathcal{M},\mathcal{M}) = H^3(\mathcal{M},\mathcal{M}) = 0$. Then $\mathcal{M}$ and $L_1(\mathcal{M})$ are *Banach-Mazur stable*, *i.e.*, $\exists \varepsilon_{\mathcal{M}} > 0$ s.t. $\forall$ vN alg. $\mathcal{N}$, TFAE:

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(iii) $L_1(\mathcal{M})$ and $L_1(\mathcal{N})$ are linearly isometric,

(iv) $d(L_1(\mathcal{M}), L_1(\mathcal{N})) < 1+\varepsilon$,

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- Which von Neumann algebras satisfy $H^2 = H^3 = 0$?

Hochschild cohomology

$\mathcal{A}$ Banach algebra and $\mathcal{X}$ Banach $\mathcal{A}$-bimodule. $\mathcal{L}^0(\mathcal{A},\mathcal{X}) = \mathcal{X}$ and $\mathcal{L}^k(\mathcal{A},\mathcal{X})$ the space of all bounded k-linear maps from $\mathcal{A}^k \to \mathcal{X}$.

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Define the 'coboundary maps' $\delta^k : \mathcal{L}^k(\mathcal{A},\mathcal{X}) \to \mathcal{L}^{k+1}(\mathcal{A},\mathcal{X})$ by:

$$\delta^0(x)(a) = ax - xa \quad \text{and}$$

$$\begin{aligned}\delta^k(\varphi)(a_1,...,a_{k+1}) = {} & a_1\varphi(a_2,...,a_{k+1}) \\ & + \textstyle\sum_{i=1}^{k-1}(-1)^i\varphi(a_1,...,a_{i-1},(a_ia_{i+1}),...,a_{k+1}) \\ & + (-1)^k\varphi(a_1,...,a_k)a_{k+1}\end{aligned}$$

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Def. :

Elements of $\operatorname{Ran}\delta^{k-1}$ are called *coboundaries*.
Elements of $\operatorname{Ker}\delta^k$ are called *cocycles*.
The kth Hochschild cohomology group is denoted:
$H^k(\mathcal{A},\mathcal{X}) = \operatorname{Ker}\delta^k / \operatorname{Ran}\delta^{k-1}$.

Rmk: restricting δ^k to completely bounded k-linear maps defines the completely bounded Hochschild cohomology $H^k_{cb}(\mathcal{A},\mathcal{X})$.

What's Hochschild cohomology good for ?

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Th. (Connes, Haagerup) :

- Let $\mathcal{A}$ be a C^*-alg., then $\mathcal{A}$ is nuclear iff $\mathcal{A}$ is amenable.
- Let $\mathcal{M}$ be a vN alg., then $\mathcal{M}$ is injective iff $H^1(\mathcal{M}, \mathcal{X}^*) = 0$ for any normal dual Banach bimodule $\mathcal{X}^*$.

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Th. (Kirchberg'96) :

A C^*-alg. $\mathcal{A}$ has the similarity property iff $H^1_\pi(\mathcal{A}, \mathbb{B}(H)) = 0$ for any bounded alg. homomorphism $\pi : \mathcal{A} \to \mathbb{B}(H)$ (here $a \cdot T = \pi(a)T$).

What's Hochschild cohomology good for ?

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A C^*-alg. $\mathcal{A}$ has the *similarity property* if for any bounded alg. homomorphism $\pi : \mathcal{A} \to \mathbb{B}(H)$, there is $S \in \mathbb{B}(H)$ s.t. $x \in \mathcal{A} \mapsto S\pi(x)S^{-1} \in \mathbb{B}(H)$ is $*$-homomorphism.

Famous question (Kadison's similarity problem '55):

Does every C^*-algebra have the similarity property ?

Th. (Kirchberg'96) :

A C^*-alg. $\mathcal{A}$ has the similarity property iff $H^1_\pi(\mathcal{A}, \mathbb{B}(H)) = 0$ for any bounded alg. homomorphism $\pi : \mathcal{A} \to \mathbb{B}(H)$ (here $a \cdot T = \pi(a)T$).

- Other reformulations:

-automatic complete boundedness (Haagerup)

-in terms of length of operator algebras (Pisier)

Vanishing of the Hochschild cohomology ?

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For any $\mathcal{M}$ von Neumann algebra, $H^1(\mathcal{M}, \mathcal{M}) = 0$.

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- The question above is open for type II_1 factors only.
- Consequence of Th.2: if one can exhibit a non Banach-Mazur stable type II_1 factor, then the answer to the question above would be no!

Vanishing of the Hochschild cohomology ?

Cor. (J. R. '20) :

Let $\mathcal{M}$ be a von Neumann algebra satisfying at least one of the following conditions:

- $\mathcal{M}_{II_1}$ has property Γ,
- $\mathcal{M}_{II_1}$ has a Cartan MASA,
- $\mathcal{M}_{II_1}\overline{\otimes}\mathcal{R} = \mathcal{M}_{II_1}$ (here $\mathcal{R}$ denotes the hyperfinite type II_1 factor).

Then for any von Neumann algebra $\mathcal{N}$ such that $d(\mathcal{M},\mathcal{N}) < 1 + 10^{-685}$, $\mathcal{M}$ and $\mathcal{N}$ are Jordan $*$-isomorphic. If moreover $\mathcal{M}$ has no summand of type I, then the bound $1 + 4 \times 10^{-183}$ is sufficient.

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Compare with:

Th. (E. Ricard, J. R.'13):

If $d_{cb}(\mathcal{M},\mathcal{N}) \leq 1 + 2.10^{-7}$, then $\mathcal{M}$ and $\mathcal{N}$ are $*$-isomorphic.

Kadison-Kastler conjecture

Def. (Kadison-Kastler distance):

Let $\mathcal{A}, \mathcal{B} \subset \mathbb{B}(H)$ be sub-C^*-alg., the Kadison-Kastler distance is the Hausdorff distance between unit balls i.e.

$$d_{KK,H}(\mathcal{A}, \mathcal{B}) = \inf \{\gamma > 0 : \forall x \in B_{\mathcal{A}}, \exists x' \in B_{\mathcal{B}}, \|x - x'\| < \gamma \text{ and } \forall y \in B_{\mathcal{B}}, \exists y' \in B_{\mathcal{A}}, \|y - y'\| < \gamma\}$$

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Conjecture (R. Kadison, D. Kastler '72) :

If two sub-C^*-alg. $\mathcal{A}, \mathcal{B}$ are close enough for $d_{KK,H}$, then they are $*$-isomorphic.

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Yes, if $\mathcal{A}$ is (Christensen *et al.*):
-an injective vN alg. or a separable nuclear C^*-alg,
-a certain crossed product, e.g. $L^\infty(0,1) \rtimes SL_3(\mathbb{Z})$. Here the key point is the cohomological condition: $H_b^2(SL_3(\mathbb{Z}), L^\infty_{\mathbb{R}}(0,1)) = 0$.

Comparing Banach-Mazur and Kadison-Kastler distance

For two C^*-alg. $\mathcal{A}, \mathcal{B}$, define

$$d_{KK}^{J}(\mathcal{A}, \mathcal{B}) = \inf_{\pi,\rho,H} \left\{ d_{KK,H}\big(\pi(\mathcal{A}), \rho(\mathcal{B})\big) \right\},$$

the infimum runs over all injective unital $*$-Jordan representations $\pi : \mathcal{A} \to \mathbb{B}(H)$, $\rho : \mathcal{B} \to \mathbb{B}(H)$ on the same Hilbert space H.

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Rmk: the missing part between Banach-Mazur distance and Kadison-Kastler distance may be related to Tingley's problem.

Proof of Th.2

Th. 2 (J. R. '20) :

$\mathcal{M}$ von Neumann algebra with predual $L_1(\mathcal{M})$. Suppose that $H^2(\mathcal{M},\mathcal{M}) = H^3(\mathcal{M},\mathcal{M}) = 0$. Then $\mathcal{M}$ and $L_1(\mathcal{M})$ are *Banach-Mazur stable*, *i.e.*, $\exists \varepsilon_{\mathcal{M}} > 0$ s.t. $\forall$ vN alg. $\mathcal{N}$, TFAE:

(i) $\mathcal{M}$ and $\mathcal{N}$ are Jordan $*$-isomorphic,

(ii) $\mathcal{M}$ and $\mathcal{N}$ are linearly isometric,

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1. "Unitize" isomorphism with small distortion.
2. Prove that a unital almost isometric bijection is almost Jordan multiplicative and almost selfadjoint.
3. Exploit the "Banach algebraic stability" of von Neumann algebras s.t. $H^2 = H^3 = 0$.

Step 1: Unitization of ismorphisms

Remember that if $T : \mathcal{M} \to \mathcal{N}$ is a surjective linear isometry, then T sends unitaries of $\mathcal{M}$ onto unitaries of $\mathcal{N}$.

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Let $\mathcal{A}$, $\mathcal{B}$ be two unital C^*-algebras. Let $T : \mathcal{A} \to \mathcal{B}$ be a linear isomorphism such that $\|T\|\|T^{-1}\| \leq 1 + \epsilon$, with $\epsilon < 1/2$. Then, for any unitary $u \in \mathcal{A}$, $T(u)$ is invertible and

$$\|T(u)^{-1}\| \leq \frac{1}{\|T\|(1 - 2\epsilon)}.$$

Consequently, there exists a unitary $u' \in \mathcal{B}$ such that

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Rmk: we can define $L : \mathcal{A} \to \mathcal{B}$ by $L(x) = T(1)^{-1}T(x)$ which is unital.

Step 2: Almost multiplicativity, almost selfadjointness

$L : \mathcal{A} \to \mathcal{B}$ is a linear map between C^*-alg, define:

· a bilinear map $L^J : \mathcal{A}^2 \to \mathcal{B}$ by $L^J(x, y) = L(x \diamond y) - L(x) \diamond L(y)$

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Rmk: In the cb-case, we would get S almost multiplicative.

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Step 2: Focus on projections

Lem. (Johnson, Kadison'75):

$\mathcal{A}$ C^*-alg. and $x \in \mathcal{A}$ selfadjoint such that $\|x^2 - x\| < 1/4$. Then $\exists$ projection q in $\mathcal{A}$ s.t. $\|x - q\| \leq 1/2 - (1/4 - \|x^2 - x\|)^{1/2}$.

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Cor. (J.R.'20):

$T : \mathcal{M} \to \mathcal{N}$ be a unital selfadjoint isomorphism s.t. $\|T\| \leq 1 + \varepsilon$ and $\|T^{-1}\| \leq 1 + \varepsilon$ with $\varepsilon < 10^{-8}$. Then for every projection p in $\mathcal{M}$, $\exists$ q projection in $\mathcal{N}$ s.t. $\|T(p) - q\| \leq 140\sqrt{\varepsilon}$.
If p is central, then q is unique and the mapping $p \in \mathcal{P}(\mathcal{Z}(\mathcal{M})) \mapsto q \in \mathcal{P}(\mathcal{Z}(\mathcal{N}))$ extends to a $*$-isomorphism from $\mathcal{Z}(\mathcal{M})$ onto $\mathcal{Z}(\mathcal{N})$.

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Proof in the cb-case: S almost multiplicative, we can define m a new multiplication $\mathcal{M}$, $\mathrm{m}(x,y) = S^{-1}\big(S(x)S(y)\big)$. Then $\| \mathrm{m} - \mathrm{m}_{\mathcal{M}} \|_{cb} \leq O(\varepsilon) < \delta$, we get $\pi = S\Phi^{-1} : \mathcal{M} \to \mathcal{N}$ $*$-isom.

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Th. (E. Christensen, E. Effros, A. Sinclair '89)

Let $\mathcal{M}$ be a vN alg. Then, $H^k_{cb}(\mathcal{M},\mathcal{M}) = 0$ for any k.

Step 3: Approximate Jordan decomposition

$T : \mathcal{M} \to \mathcal{N}$ and $p \in \mathcal{M}$, $q \in \mathcal{N}$ projections, denote:
· $T_p^q : p\mathcal{M}p \to q\mathcal{N}q$ defined by $T_p^q(x) = qT(pxp)q$
· $L^\vee(x,y) = L(xy) - L(x)L(y)$ and $L^\wedge(x,y) = L(xy) - L(y)L(x)$.

Th. (J.R.'20)

Let $T : \mathcal{M} \to \mathcal{N}$ be a unital selfadjoint isomorphism such that $\|T\| \leq 1+\varepsilon$ and $\|T^{-1}\| \leq 1+\varepsilon$ with $\varepsilon < 5 \times 10^{-13}$. Suppose that the type I part of $\mathcal{M}$ admits homogeneous von Neumann subalgebras of even degree only.
Then *exists* central projections $p \in \mathcal{M}$, $q \in \mathcal{N}$ s.t. $\max\{\|(T_p^q)^\vee\|, \|(T_{1-p}^{1-q})^\wedge\|\} \leq 3147585\sqrt{\varepsilon}$ and $\|T - (T_p^q \oplus T_{1-p}^{1-q})\| \leq 1484\sqrt{\varepsilon}$.

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Rmk: ultraproduct argument...?

Concluding remarks

Idea: "vanishing of cohomology implies stability". How about the converse ?

Def.

$\mathcal{M}$ is *strongly Kadison-Kastler stable* if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for any injective normal $*$-homomorphism $\rho : \mathcal{M} \to \mathbb{B}(H)$, for any vN subalgebra $\mathcal{N} \subset \mathbb{B}(H)$, $d_{KK,H}(\rho(\mathcal{M}), \mathcal{N}) < \delta$ implies $\exists U \in \mathbb{B}(H)$ unitary s.t. $U^* \rho(\mathcal{M}) U = \mathcal{N}$ and $\|U - I_H\| < \varepsilon$.

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Th. (Christensen et al.'14)

Every strongly Kadison-Kastler stable II_1-factor has the similarity property (i.e. $H^1_\pi(\mathcal{A}, \mathbb{B}(H)) = 0$ for any bounded alg. hom. $\pi : \mathcal{A} \to \mathbb{B}$).

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THANK YOU!