Projection theorem of a discrete-time quantum walk

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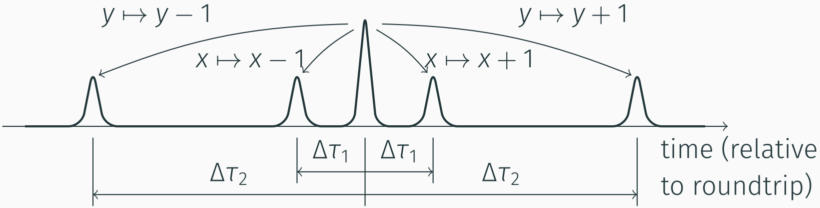
Outline

1. Motivation
2. Projection theory
3. Examples
4. Undoing projection
5. Conclusions

Motivation

Experimental 2D walk

2D quantum walk in a time-multiplexed optical implementation (Schreiber et al. 2012):



$\Delta\tau_1$ and $\Delta\tau_2$ chosen such that $x \cdot \Delta\tau_1 + y \cdot \Delta\tau_2 + t \cdot \tau_{\text{round}}$ distinct $\forall x, y, t$

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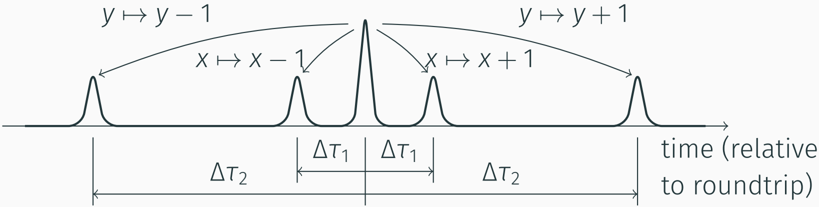

What happens if $\Delta\tau_2$ and $\Delta\tau_1$ are the same (but still $\ll \tau_{\text{round}}$)?

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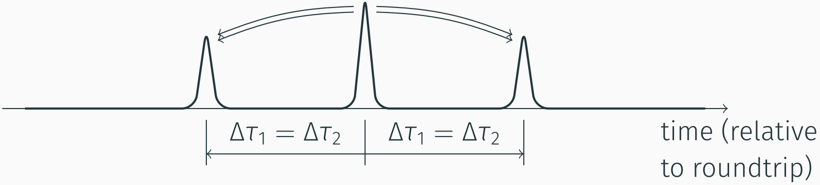

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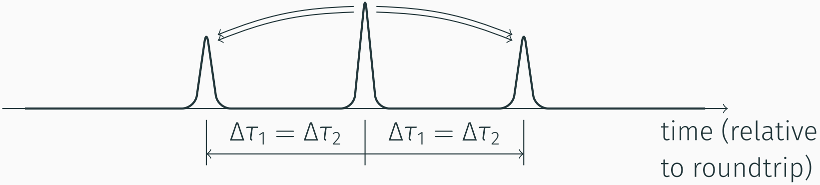

What happens if $\Delta\tau_2$ and $\Delta\tau_1$ are the same (but still $\ll \tau_{\text{round}}$)?

- points of the same $x + y$ in the same roundtrip overlap
  - forced interference, observed amplitude = sum of original amplitudes of different lattice points
  - conservation of energy (optics) $\hat{=}$ unitarity (QW)

Dynamics

2D quantum walk, constant coin:

$$\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_C = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^4$$

$$C = \mathbb{I}_P \otimes C_0, \quad C_0 \in \mathcal{U}(4)$$

$$S = \sum_{x,y} \big(|x+1,y\rangle \langle x,y| \otimes \Pi_R + |x-1,y\rangle \langle x,y| \otimes \Pi_L + \\ |x,y+1\rangle \langle x,y| \otimes \Pi_U + |x,y-1\rangle \langle x,y| \otimes \Pi_D \big)$$

Dynamics

2D quantum walk, constant coin – recurrence relation:

$$|\psi(t)\rangle = \sum_{x,y} |x, y\rangle \otimes |\alpha^{(t)}_{x,y}\rangle \qquad (|\alpha^{(t)}_{x,y}\rangle \in \mathbb{C}^4) :$$

$$|\alpha^{(t+1)}_{x,y}\rangle = \Pi_R C\,|\alpha^{(t)}_{x-1,y}\rangle + \Pi_L C\,|\alpha^{(t)}_{x+1,y}\rangle + \Pi_U C\,|\alpha^{(t)}_{x,y-1}\rangle + \Pi_D C\,|\alpha^{(t)}_{x,y+1}\rangle$$

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Corresponds to a different quantum walk:

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Corresponds to a different quantum walk:

$$\mathcal{H}' = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^4 =: \mathcal{H}_{P'} \otimes \mathcal{H}_C$$ ← QW on a line, 4D coin space

$$C' = \mathbb{I}_{P'} \otimes C_0$$ ← coin matrix unchanged

$$S' = \sum_z \big(|z+1\rangle\langle z| \otimes (\Pi_R + \Pi_U) + |z-1\rangle\langle z| \otimes (\Pi_L + \Pi_D) \big)$$ ← two basis coin states to each direction

Projection theory

Notation

X ... set of positions

Γ ... set of displacements: injections $X \to X$
For a $c \in \Gamma$, we denote $x \cdot c := c(x)$

$\mathcal{H}_P = \ell^2(X)$ s.t. $\{|x\rangle \mid x \in X\}$ ON basis, similarly $\mathcal{H}_C = \ell^2(\Gamma)$,
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$C, S \in \mathcal{U}(\mathcal{H})$:

$C = \mathbb{I}_P \otimes C_0, \quad C_0 \in \mathcal{U}(\mathcal{H}_C)$ (for now)

$S = \sum_{x \in X} \sum_{c \in \Gamma} |x \cdot c\rangle \langle x| \otimes |c\rangle \langle c| : \; |x\rangle |c\rangle \mapsto |x \cdot c\rangle |c\rangle$

Given $|\psi_0\rangle \in \mathcal{H}$: $\quad |\psi_n\rangle := (SC)^n |\psi_0\rangle$

Projection map

Projection map: any function ρ mapping X onto another set X' s.t.

$$\forall x, y \in X, \forall c \in \Gamma,\ \rho(x) = \rho(y) \Leftrightarrow \rho(x \cdot c) = \rho(y \cdot c)$$

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ρ surjective $\Rightarrow X/\sim_\rho \cong X'$, we identify their elements

Projection operator

Given X, Γ, C_0, X', $\rho : X \to X'$, we construct

$$\begin{aligned} \mathcal{H}'_P &= \ell^2(X'), \\ \mathcal{H}'_C &= \mathcal{H}_C, \\ \mathcal{H}' &= \mathcal{H}'_P \otimes \mathcal{H}'_C \end{aligned}$$

and

$$\$: \mathcal{H} \to \mathcal{H}' : |x\rangle\,|c\rangle \mapsto |\rho(x)\rangle\,|c\rangle$$

Projection operator

$$\$: \mathcal{H} \to \mathcal{H}' : |x\rangle\,|c\rangle \mapsto |\rho(x)\rangle\,|c\rangle$$

Now

$$C' = \mathbb{I}_{P'} \otimes C_0,$$
$$S' = \sum_{x' \in X'} \sum_{c \in \Gamma} |c'(x')\rangle\,\langle x'| \otimes |c\rangle\,\langle c|$$

satisfy

$$S'\$ = \$S,$$
$$C'\$ = \$C$$

and as such, for a given $|\psi_0\rangle \in \mathcal{H}$,

$$|\psi_0'\rangle := \$\,|\psi_0\rangle\,,$$
$$|\psi_n'\rangle := (S'C')^n\,|\psi_0'\rangle = (S'C')^n\$\,|\psi_0\rangle = \$(SC)^n\,|\psi_0\rangle = \$\,|\psi_n\rangle\,.$$

Limits of validity

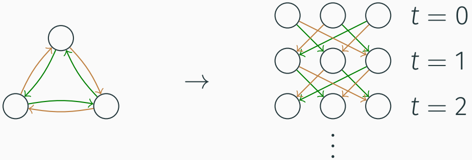
In general, $\$$ is non-injective and unbounded.

That is: states $|\psi\rangle$ can be found for which $\$ |\psi\rangle$ is undefined or zero.

For all other cases, a quantum walk (as a sequence of states) is mapped to a quantum walk, $\$ |\psi\rangle$ is normalizable and $U' = S'C'$ preserves this norm.

Generalizations

1. Position-dependent coins:
   Projection by ρ is possible *iff* the coin matrix is constant over equivalence classes of $\sim_\rho$.
2. Time-dependent coins:
   Can be treated as a special case of position-dependent coins with a layered, directed graph.



3. Phases:
   \$ can in principle map $|x, c\rangle$ to $e^{i\phi(x)} |\rho(x), c\rangle \rightarrow$ twisted PBCs etc.

Specializations

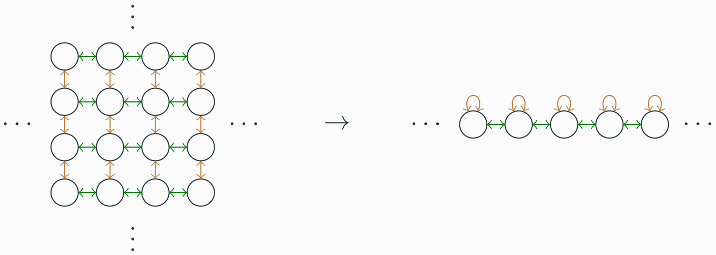
Often in practice:

- X is a group,
- $\Gamma \subset X$ with $c \in \Gamma : c(x) =: x \cdot c$,
- $X' = H \backslash X$ where $H \leq G$ (the set of right posets of H in X),
- $\rho : x \mapsto Hx$.

Examples

Lazy quantum walk

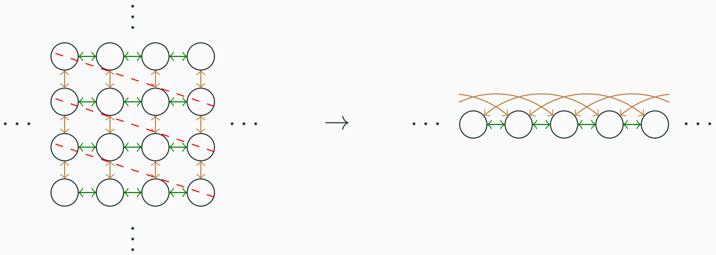
$$X = \mathbb{Z}^2,\ \Gamma = \{(\pm 1, 0), (0, \pm 1)\},\ X' = \mathbb{Z},\ \rho(x, y) = x \quad \dots \quad |\beta_x\rangle = \sum_{y \in \mathbb{Z}} |\alpha_{x,y}\rangle$$



Amplitudes of a 2D walk summed along an axis → a lazy QW on a line (with two lazy coin states)

Quantum walk with jumps

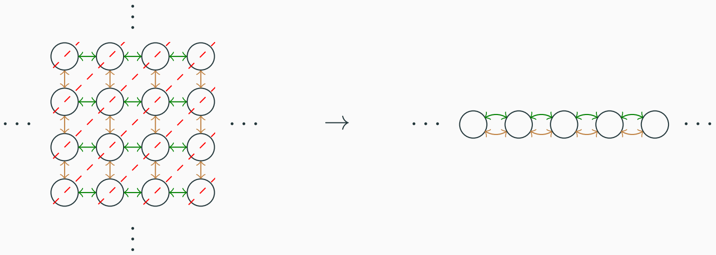
$X = \mathbb{Z}^2,\ \Gamma = \{(\pm 1, 0), (0, \pm 1)\},\ X' = \mathbb{Z},\ \rho(x, y) = x + ky,\ k \in \mathbb{Z}$



Amplitudes of a 2D walk summed along an oblique line $\rightarrow$ a QW on a line with coherent jumps of length k

Double line quantum walk

$X = \mathbb{Z}^2,\ \Gamma = \{(\pm 1, 0), (0, \pm 1)\},\ X' = \mathbb{Z},\ \rho(x, y) = x - y$



Case of $k = 1$: a walk on a line with doubled edges (Lorz et al. 2019)

Projections of 2D QW

Example of an immediate result:

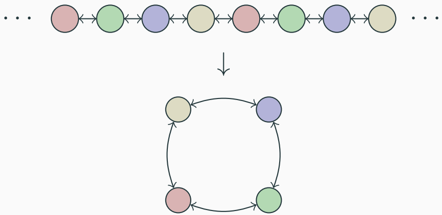
All of these settings:

- lazy quantum walk on a line with two lazy coin states,
- quantum walk on a line with coherent jumps of fixed size,
- quantum walk on a line with double edges

trivially allow trapped states (i.e., finitely supported eigenstates of evolution operator) with a Grover coin matrix.

QW on a circle

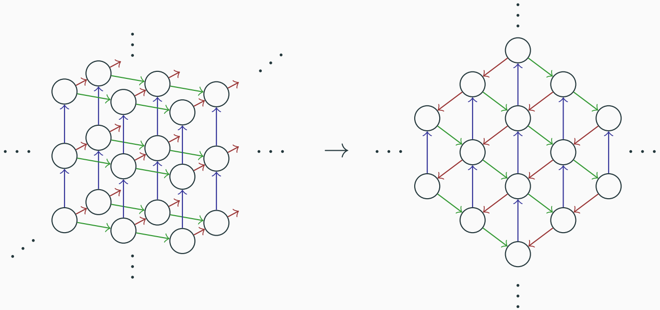
$$X = \mathbb{Z},\ \Gamma = \{\pm 1\},\ X' = \mathbb{Z}_{\mathbb{N}},\ \rho(x) = [x]_{\text{mod } N} \quad \dots \quad |\beta_x\rangle = \sum_{y\in\mathbb{Z}} |\alpha_{x+Ny}\rangle$$



Amplitudes of a 1D walk summed over residue classes modulo $N \rightarrow$ QW on a N-circle (twist phase possible in a generalization)

Triangular lattice

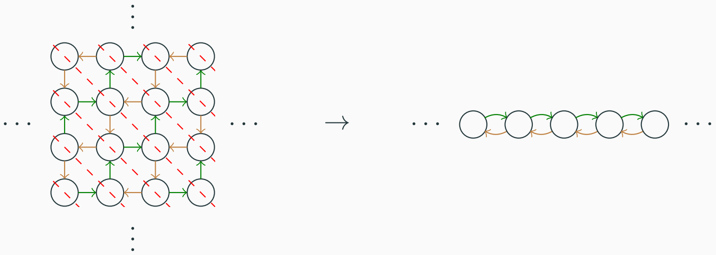
$X = \mathbb{Z}^3,\ \Gamma = \{+e_1, +e_2, +e_3\},\ X' = \mathbb{Z}^2,\ \rho(x, y, z) = (x - z, y - z)$



QW on a directed 3D lattice projected along a space diagonal $\to$ QW on a directed triangular lattice

L-lattice $\to$ line

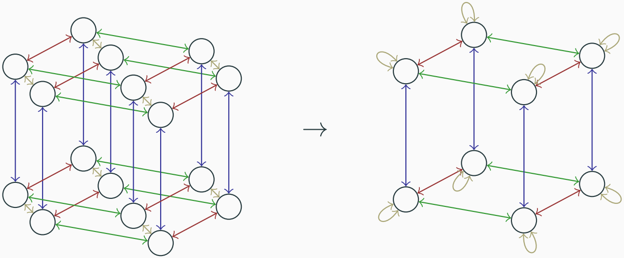
$$X = \langle a, b \mid a^2 b^{-2} = e \rangle,\ \Gamma = \{a, b\},\ X' = \mathbb{Z},\ \rho(a) = -\rho(b) = 1$$



QW on an L-lattice projected along diagonal antiparallel to the coin direction assignment $\to$ a QW on a line

Hypercube $\to$ hypercube with self-loops

$$X = \mathbb{Z}_2^{\times d},\ \Gamma = \{\cdots\},\ X' = \mathbb{Z}_2^{\times(d-1)},\ \rho(\overline{a_1 \cdots a_{d-1} a_d}) = \overline{a_1 \cdots a_{d-1}}$$



QW on a d-dimensional hypercube $\to$ QW on a $(d-1)$-dimensional hypercube with self-loops (Potoček et al. 2009)

Undoing projection

Simplest example

Dimensional reduction of 2D to 1D lattice:

Let $X = \ell^2(\mathbb{Z}^2)$, $X' = \ell^2(\mathbb{Z})$, $\rho : X \to X' : \rho(x, y) = kx + ly$ with $k, l \in \mathbb{Z}$, $\gcd(k, l) = 1$.

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Then

$$M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \rho(x, y) \\ \sigma(x, y) \end{pmatrix} \quad \text{and thus} \quad M^{-1} \begin{pmatrix} \rho(x, y) \\ \sigma(x, y) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Phase-enhanced projection

We can use $\sigma(x, y)$ to "tag" different points by their phase:

$$\$_\phi : \mathcal{H} \to \mathcal{H}' : |x, y\rangle |c\rangle \mapsto e^{i\phi\sigma(x,y)} |\rho(x, y)\rangle |c\rangle$$

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and thus for a generic $|\psi\rangle \in \mathcal{H}$, $\forall x, y \in \mathbb{Z}$,

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We can completely reconstruct $|\psi\rangle$ if projections $\$_\phi |\psi\rangle$ are known for a.e. $\phi \in (0, 2\pi)$.

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The operator Φ can be absorbed into C' or S'.

Thus, accessing projections for different angles $\phi \triangleq$ altering relative phases in coin matrix.

Experimentally (example from Motivation): alternatively by introducing sub-period differences between $\Delta\tau_1$ and $\Delta\tau_2$.

Conclusions

Conclusions

- A QW on a simple structure with a coin of higher dimension (as seen e.g. in lazy walks):
    - hints at existence of an "ancestral" quantum walk on a higher-dimensional structure
    - if so, we can probably explain its dynamics by solving the latter
- A projected walk not necessarily "simpler" than the original
    - projections of Euclidean lattices can be more interesting graphs
- Reversible in some cases
    - a smaller structure can teach us about a larger one
    - a fundamentally 1D experiment can give enough data to reconstruct 2D evolution
- Other similar but *independent* techniques:
    - tensor product decomposition
    - utilizing permutation symmetries
    - These can be combined in understanding QWs!