The gambler's ruin problem and quantum measurement

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Quantum Mechanics (QM)

QM needs two ingredients:

- Dynamical law for the unitary evolution of a state (*e.g.* Schrödinger equation) of an isolated system
- Law for the probabilities of apparently random measurement: Born rule

Well known problem:

Deduce the very possibility of measurement, their apparent random nature and the Born rule from the unitary evolution of isolated systems

What is known

- Many attempts to solve the problem
- The only thing thing all physicists agree upon is decoherence
- But does decoherence tell the whole story? No agreement!

Decoherence in a nutshell

- Quantum system S and its environment E (macroscopic, finite temperature etc.)

- Measurement $=$ interaction between S and E

- S is best described by its reduced density operator ρ^S

- There is a basis in which the interaction makes ρ^S become diagonal *i.e.*

- The interaction between S and E transforms a quantum superposition into a classical one (no interference)

Reformulation of the problem

- But it seems the problem remains because

  We do not observe classical superpositions!

- Two standard attitudes:
  - There is no problem. Decoherence tells the whole story
  - Part of the problem remains and something must be added to decoherence

Decoherence tells the whole story

- Observers also get split into classical superpositions
- Measurements happen in the minds of observers
- Seems to warrant a many world interpretation of QM

  Problems with this point of view:

- Ocam razor
- Theory seems to necessitate a non trivial interpretation and there is yet no **physical model** of what we experience. Will come with a model of the brain?

Decoherence does not tell the whole story

- OK, but then what?

  Constraints are

- Consistency with deterministic unitary evolution of isolated system

- Consistency with decoherence

- Phenomenology: random nature of measurements + Born rule

- Hope: no interpretation necessary *i.e.* quantum measurement can be modelled as any other physical phenomenon

This talk

- Classical example: the Langevin particle
- Quantum measurement: two key points
- General unbiased quantum measurement
- Example
- Discussion
- Bonus: Discrete geometry from quantum walks (Debbasch, 2019)

The Langevin particle

- Non quantum particle S diffusing through collisions in a non quantum fluid E
- Complete description (CD) : dynamical equations for the positions and momenta of all particles (S and E). Standard Hamiltonian classical mechanics.
- Useless in practice $\Rightarrow$
- Effective description (ED): dynamical equations for the position and momentum of S alone
- Because of random collisions, the momentum of S undergoes stochastic jumps (random in time and amplitude)
- Thus, ED is stochastic though CD is not

The Langevin particle

- Assumptions of E (*e.g.* equilibrium) and S ($m_S >> m_E$) $\Rightarrow$

  On long enough time-scales

$$
\begin{aligned}
dx_t &= \frac{p_t}{m_S} dt \\
dp_t &= -\alpha \, p_t \, dt + D \, dB_t
\end{aligned}
$$

- The statistical averages obey the deterministic equations

$$
\begin{aligned}
d < x >_t &= \frac{< p >_t}{m_S} dt \\
d < p >_t &= -\alpha \ < p >_t \ dt
\end{aligned}
$$

The Langevin particle

- On average, the momentum goes to zero on a time-scale α^{-1}
- But there are fluctuations $\Rightarrow$
  - Diffusion of S even for times $>> \alpha^{-1}$
  - Thermalization (fluctuation-dissipation theorem)
- Looking only at averages, you miss a lot of physics!

Quantum measurement: two key points

Consider a single quantum system S interacting with an environment E

According to QM, in the Schrödinger picture

- The instantaneous state $| \Psi >$ of $S \cup E$ can be described its wave-function Ψ

$$i\hbar \, \partial_t \Psi = H \Psi$$

- Alternately, the same state can be described by $\rho = | \Psi >< \Psi |$

$$\partial_t \rho = i\hbar \, [H, \rho]$$

- No statistical physics yet!

Quantum measurement: Key point 1

- As in non quantum physics, the above approach is not tractable $\Rightarrow$
- Effective description of the dynamics of S
- Right variable $=$ reduced density operator of S

  $\rho^S = \mathrm{Tr}_E\, \rho$

- Key point 1

  ρ^S is a stochastic variable

- This is well-known from decoherence, but rarely stated explicitly

Quantum measurement: Key point 1

- Decoherence can be ‘rapid’ or ‘slow’
- Slow decoherence $\rightarrow$ Stochastic Differential Equation (SDE) for ρ^S
- Rapid decoherence $\rightarrow$ no SDE for ρ^S (though ρ^S is stochastic)
- For slow and rapid decoherence (and some assumptions on E), $< \rho^S >$ obeys a PDE sometimes called Master Equation (ME)
- For slow decoherence, ME $=$ average of SDE obeyed by ρ^S

Quantum measurement: Key point 2

- $\rho = \rho^S \otimes \rho^E + \rho^e$

  $\rho^E =$ reduced density operator of E

  $\rho^e =$ entanglement density operator

- $\text{Tr}_E\, \rho^e = 0$

  $\text{Tr}_S\, \rho^e = 0$

- At any given time t, the information encoded in $\rho^e(t)$ is not encoded in $\rho^S(t)$ or $\rho^E(t)$

Quantum measurement: Key point 2

- ρ^S obeys the exact dynamical equation

  $$d\rho^S_t = i\hbar\left(\mathsf{Tr}_E\left[H, \rho^S \otimes \rho^E\right]\right)dt + i\hbar\left(\mathsf{Tr}_E\left[H, \rho^e\right]\right)dt$$

- First term on the left is linear in ρ^S

- Second term does not generally vanish and is not linear in ρ^S

- Key point 2

  The evolution of ρ^S may not be linear

General unbiased quantum measurement

- Focus on the diagonal components of ρ^S in the decoherence basis

  Notation: ρ^i, $1 \leq i \leq N$

- Each ρ^i is a stochastic process, starts at ρ^i_0

- Two constraints on the N processes

  $\forall t, \sum_i \rho^i_t = 1$

  $\forall t, \forall i, 0 \leq \rho^i_t \leq 1$

General unbiased quantum measurement

- The random evolution is unbiased $\Rightarrow$

  $\forall t, \forall i$, if ρ_t^i has a certain probability to increase by the amount δ, it has the same probability to decrease by the same amount δ

- Consequence

  $\forall i$, the process ρ_t^i stops when ρ_t^i reaches 0 or 1

- In mathematical language

  Each process ρ_t^i is a martingale

  The time τ^i at which a given ρ_t^i reaches 0 or 1 is a stopping time

  The time τ^i is random

Gambler's ruin problem

Translation as a gambler's ruin problem

- N gamblers
- Total fortune $= 1$
- Each gambler starts with an initial fortune ρ_0^i
- Money can only be exchanged between gamblers, not created
- The game is fair
- Each gambler stops playing when she has no more money
- Winner takes all !

Optional stopping theorem

- Several versions
- Martingale ρ with stopping time τ
- Under some 'mild' conditions (*e.g.* for a positive martingale, finite expectancy $E(\tau)$ of the stopping time)

  $E(\rho_\tau) = E(\rho_0)$

- Not trivial because τ is random!

Application to quantum measurement

- $\forall i,\ E(\rho^i_{\tau^i}) = E(\rho^i_0)$
- $E(\rho^i_{\tau^i}) = 1 \times p^i_w + 0 \times (1 - p^i_w)$

  p^i_w = probability that the gambler i wins the game
- $E(\rho^i_0) = \rho^i_0$
- $\Rightarrow p^i_w = \rho^i_0$ Born's rule

Example

- N gamblers
- Discrete fortunes: $\Delta\rho = 1/N_0$, N_0 =integer

  Say, total fortune = 1 Euro and $N_0 = 100$
- Initial fortunes $\rho_0^i = N_0^i \Delta\rho$, $0 \leq N_0^i \leq N_0$

  Say $N = 3, N_0^1 = 30, N_0^2 = 50, N_0^3 = 20$
- Gamblers play in succession by pairs, whenever and with whomever they want (no specific order or regularity)
- Each gambler of the currently playing pair rolls a dice. The gambler with the highest score receives from the other one the fortune $\Delta\rho$.
- Once the fortune of a gambler reaches 0, the gambler stops playing

Example

- Exact direct computation possible because
- During each phase, each player performs a symmetric random walk in fortune space
- $P(N_0^i)$ = probability for the random walk, starting at $\rho_0^i = N_0^i \Delta\rho$, to reach 1 before 0
- $P(N_0) = 1$, $P(0) = 0$

Example

$\forall N_0^i$

- $P(N_0^i) = \frac{1}{2}\left(P(N_0^i+1) + P(N_0^i-1)\right)$
- $P(N_0^i+1) - P(N_0^i) = P(N_0^i) - P(N_0^i-1) = \ldots = P(1) - P(0) = P(1)$
- $P(N_0^i+1) - P(1) = \sum_{k=1}^{N_0^i}(P(k+1) - P(k)) = N_0^i P(1)$
- $P(N_0^i+1) = (N_0^i+1)P(1)$
- $P(N_0) = 1 = N_0 P(1) \Rightarrow P(1) = 1/N_0$
- $P(N_0^i) = N_0^i P(1) = N_0^i/N_0 = \rho_0^i$ Born's rule

Discussion

- $\{\text{No bias}\} + \{\forall t, \forall i, \rho_t^i \geq 0\} + \{\forall t, \mathrm{Tr}\rho_t^S \text{ constant}\} \Rightarrow$ non-linearity

  Non-linearity $\Leftarrow \rho^e$

  Entanglement is responsible for both decoherence and the apparent collapse

- Born's rule is very robust because the optional stopping theorem is

- There may be noise on the off-diagonal components of ρ^S, at least for slow decoherence

  Consequences?

Discussion

- Does God play dice?

  No, because God knows everything and takes no trace!

- Physicists do not know everything, take traces, and witness apparent stochastic collapses

- Two physicists P_1 and P_2, with environments E_1 and E_2 observe S

  P_1 observes S collapse because of its interaction with some degrees of freedom in E_1

  If the same degrees of freedom also belong to E_2, then P_2 witnesses the same collapse *i.e.* P_1 and P_2 share the same 'reality'.

Next steps

- Build detailed models where the noise can be computed from micro-physics
- Perform experiments to detect the noise and/or monitor the stochastic evolution of ρ^S
- Extend to QFT
- Extend to a special and general relativistic theory of measurement