Right-angled Coxeter groups commensurable to right-angled Artin groups

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joint with
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$$W_\Gamma = \langle S \mid s^2 = 1 \textit{ for } s \in S, st = ts \textit{ for } (s,t) \in E \rangle$$

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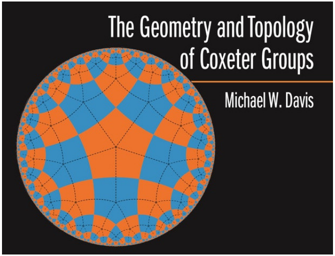
We now see that many RACGs are *not* quasi-isometric (and therefore not commensurable) to any RAAG.

Example: some hyperbolic reflection groups

Γ an n-cycle, $n \geq 5$. The RACG W_Γ is a Fuchsian group.



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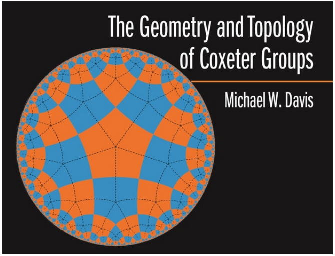

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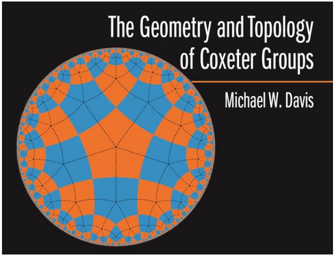

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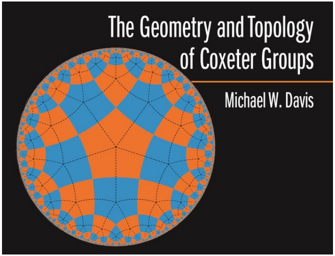

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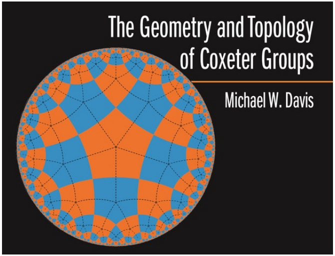

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Every one-ended RAAG contains $\mathbb{Z}^2$ subgroups and are not hyperbolic. So W_Γ is not quasi-isometric to any RAAG.

More RACGs not QI to RAAGs

Divergence

Divergence is a quasi-isometry invariant measuring the max rate a pair of geodesic rays diverge in the Cayley graph of a group.

A RAAG has either linear, quadratic or exponential divergence. [Behrstock-Charney]

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Morse boundary

The *Morse boundary* is a quasi-isometry invariant which is a boundary associated to Morse geodesics.

A RAAG has totally disconnected Morse boundary. [Charney-Sultan, Cordes-Hume, Charney-Cordes-Sisto]

There are RACGs with quadratic divergence and Morse boundary with non-trivial connected components. [Behrstock]

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To prove the above theorem, we find a way to detect finite-index RAAG subgroups of RACGs.

Candidate RAAG subgroups: visual RAAGs

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Visual RAAGs were first studied in LaForge's thesis.

Visual RAAG Example

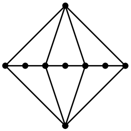

Figure 1: RACG defining graph Γ

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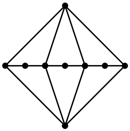

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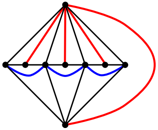

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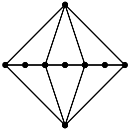

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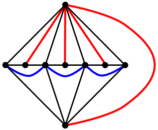

Figure 2: Choice of $\Lambda \subset \Gamma^c$ in red and blue. Λ has two components in this case.

We always draw Γ in black and Λ in colors. Each color of Λ corresponds to a component of Λ.

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We will now discuss necessary conditions on Λ for G_Λ to be a visual RAAG. The first three are due to LaForge.

These conditions naturally are characterized by the number of components of Λ involved.

Single component conditions: $\mathcal{R}_1$ and $\mathcal{R}_2$

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Given a path in Λ with endpoints p and q, then p and q do not span an edge of Γ.

$\mathcal{R}_2$ is necessary

Lemma (LaForge)

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As RAAGs are torsion-free, G_Λ cannot be a RAAG. □

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Definition ($\mathcal{R}_3$)

Let Λ_c and Λ_d be components of Λ. Let c_1, d_1, c_2, d_2 be a square in Γ with $c_1, c_2 \in \Lambda_c$ and $d_1, d_2 \in \Lambda_d$. Then Γ contains the join of $\mathrm{Hull}_\Lambda\{c_1, c_2\}$ and $\mathrm{Hull}_\Lambda\{d_1, d_2\}$.

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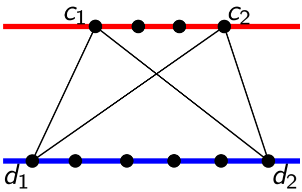

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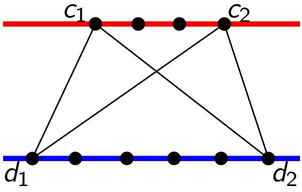

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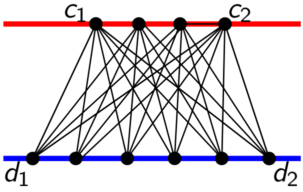

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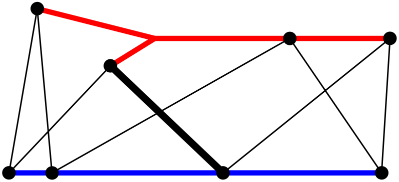

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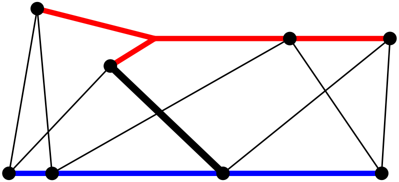

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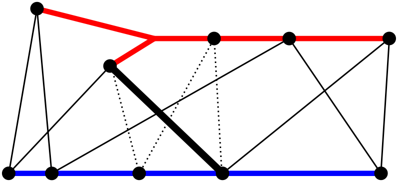

Characterization when Λ has at most two components

Theorem (Dani-L)

Suppose $\Lambda \subset \Gamma^c$ has at most two components.

$(G_\Lambda, E(\Lambda))$ *is a RAAG system* $\iff$ $\mathcal{R}_1 - \mathcal{R}_4$ *are satisfied.*

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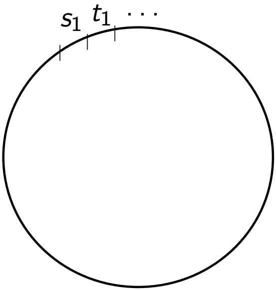
For a contradiction, suppose w is trivial in G_Λ and w is a non-trivial, minimal length word in the RAAG A_Δ.

Form the disk diagram in W_Γ with boundary w:



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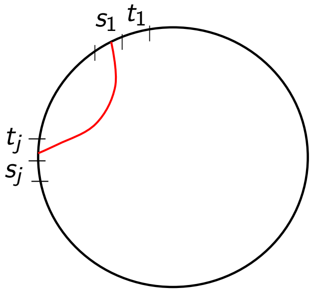

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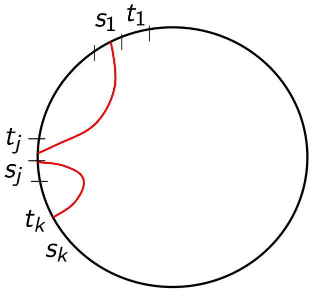

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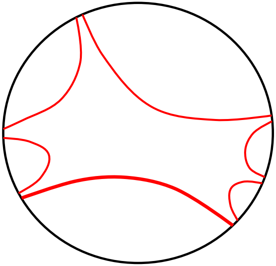

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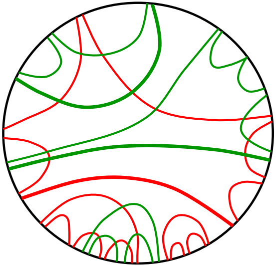
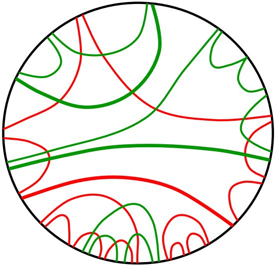


The disk diagram has the structure of *hyperplane chains*.

Hyperplane chains



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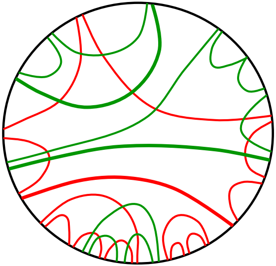

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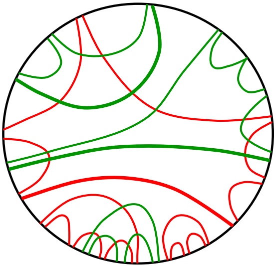

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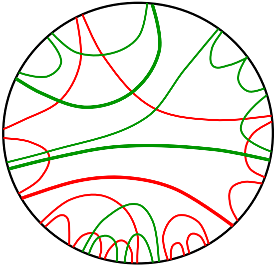

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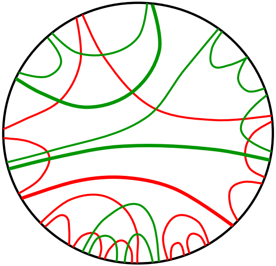

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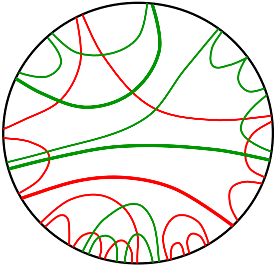

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- The proof then uses conditions $\mathcal{R}_1 - \mathcal{R}_4$ and the structure of hyperplane chains to perform "moves", corresponding to relations in A_Δ, producing diagrams of "lower complexity."

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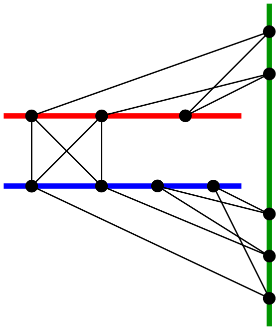

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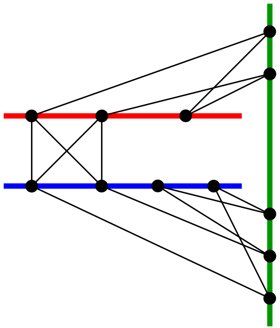

then Γ *must* contain a triangle.

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$\mathcal{F}_2$: Given vertices s and t in difference components of Λ, then there is a path in Γ from s and t containing only vertices in these components.

Application

Recall our goal theorem:

Theorem (Dani-L)

Let W_Γ be a 2-dimensional, one-ended RACG with Γ planar.
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By the previous theorem, G_Λ is a RAAG commensurable to W_Γ.

Thank you!