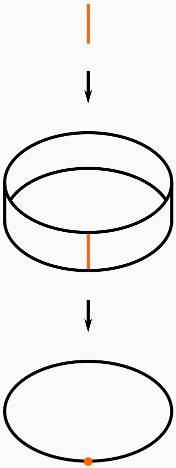
Computing fibring of 3-manifolds and free-by-cyclic groups

Dawid Kielak

3-Manifolds

Fibring (over the circle)

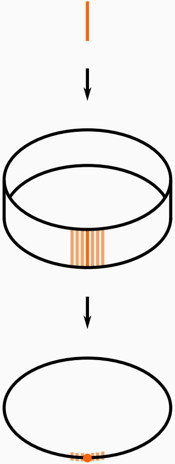


Interval
$[0, 1]$

Annulus
$\mathbb{S}^1 \times [0, 1]$

Circle
$\mathbb{S}^1$

Fibring (over the circle)

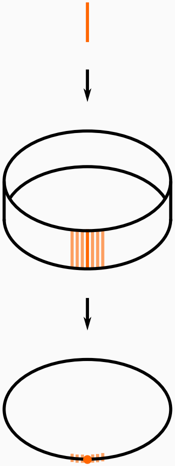


Interval
$[0, 1]$

Annulus
$\mathbb{S}^1 \times [0, 1]$

Circle
$\mathbb{S}^1$

Fibring (over the circle)



Interval
$[0,1]$

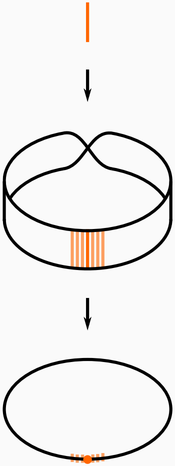
Annulus
$\mathbb{S}^1 \times [0,1]$

Circle
$\mathbb{S}^1$

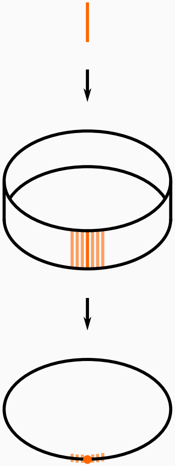
Interval
$[0,1]$

Möbius band

Circle
$\mathbb{S}^1$



Fibring (over the circle)



Interval
$[0, 1]$

Annulus
$\mathbb{S}^1 \times [0, 1]$

Circle
$\mathbb{S}^1$

Surface

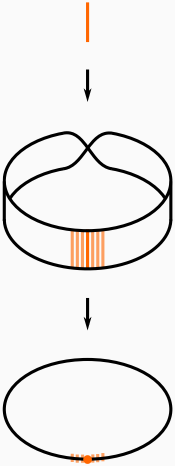
3-Manifold

$\mathbb{S}^1$

Interval
$[0, 1]$

Möbius band

Circle
$\mathbb{S}^1$



Fibring formally

Definition

Let X be a connected topological space. A continuous function $f\colon X \to \mathbb{S}^1$ is a *fibring* if and only if for every $p \in \mathbb{S}^1$ there exists a neighbourhood U of p such that $f^{-1}(U) \cong f^{-1}(p) \times U$, where the homeomorphism respects the map f.

Fibring formally

Definition

Let X be a connected topological space. A continuous function $f\colon X \to \mathbb{S}^1$ is a *fibring* if and only if for every $p \in \mathbb{S}^1$ there exists a neighbourhood U of p such that $f^{-1}(U) \cong f^{-1}(p) \times U$, where the homeomorphism respects the map f.

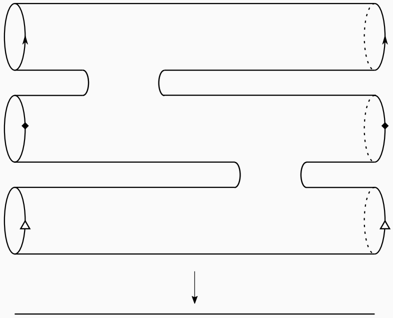
Non-example

Take a surface Σ of genus $\geqslant 2$. Given any map $\Sigma \to \mathbb{S}^1$, there will be various homeomorphism types of fibres.

Fibring formally

Non-example

Take a surface Σ of genus $\geqslant 2$. Given any map $\Sigma \to \mathbb{S}^1$, there will be various homeomorphism types of fibres.

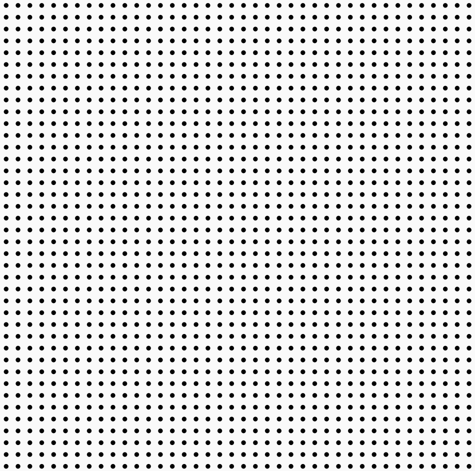


How do 3-manifolds fibre?

Theorem (Thurston 1986)

There exists a polytope P_M controlling which $M \to \mathbb{S}^1$ fibres.

- Every dot is a map $M \to \mathbb{S}^1$

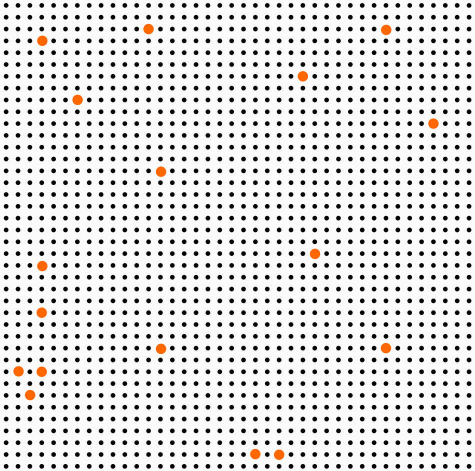


How do 3-manifolds fibre?

Theorem (Thurston 1986)

There exists a polytope P_M controlling which $M \to \mathbb{S}^1$ fibres.

- Every dot is a map $M \to \mathbb{S}^1$
- A dot is a fibring $\Leftrightarrow$ it is orange

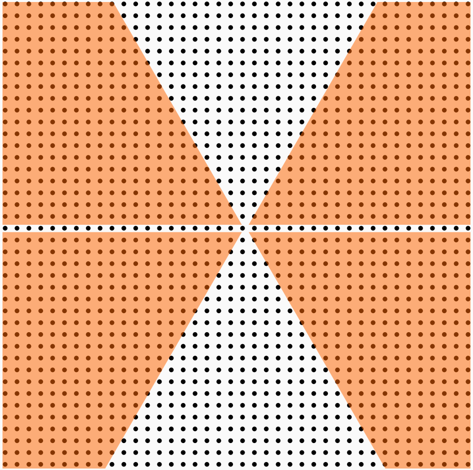


How do 3-manifolds fibre?

Theorem (Thurston 1986)

There exists a polytope P_M controlling which $M \to \mathbb{S}^1$ fibres.

- Every dot is a map $M \to \mathbb{S}^1$
- A dot is a fibring $\Leftrightarrow$ it lies in the orange field

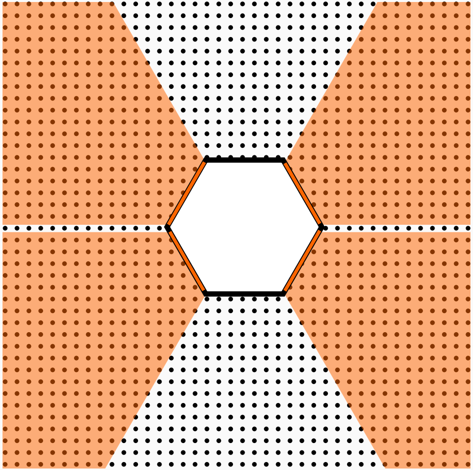


How do 3-manifolds fibre?

Theorem (Thurston 1986)

There exists a polytope P_M controlling which $M \to \mathbb{S}^1$ fibres.

- Every dot is a map $M \to \mathbb{S}^1$
- A dot is a fibring $\Leftrightarrow$ it lies in the orange field
- The orange field is the cone over some faces of P_M



Is it practical?

Theorem (Tollefson–Wang)

Under extremely mild conditions on M, there is an algorithm computing P_M. The input is a triangulation of M.

Theorem (Schleimer, Cooper–Tillmann)

Under the same conditions, there is an algorithm computing the fibred faces.

There is even a Sage package! [Worden]

Free-by-cyclic groups

Enter the group theory

What does a fibring 3-manifold look like algebraically?

Short exact sequences

$$\Sigma \to M \to \mathbb{S}^1$$

'short exact sequence'

Enter the group theory

What does a fibring 3-manifold look like algebraically?

Short exact sequences

$$\Sigma \to M \to \mathbb{S}^1$$

'short exact sequence'

$$\pi_1(\Sigma) \to \pi_1(M) \to \mathbb{Z}$$

an honest short exact sequence

Enter the group theory

What does a fibring 3-manifold look like algebraically?

Short exact sequences

$$\Sigma \to M \to \mathbb{S}^1$$

'short exact sequence'

$$\pi_1(\Sigma) \to \pi_1(M) \to \mathbb{Z}$$

an honest short exact sequence

So $G = \pi_1(M) = \pi_1(\Sigma) \rtimes \mathbb{Z}$, a surface-by-cyclic group.

Free-by-cyclic groups

When M has boundary, so does Σ, and so $\pi_1(\Sigma) = F_n$.

The converse is not true:

Important fact

Not every free-by-cyclic group $F_n \rtimes \mathbb{Z}$ is a 3-manifold group!

The two families are closely related.

Back to fibring

Theorem (Stallings)

A map $f\colon M \to \mathbb{S}^1$ is homotopic to a fibring if and only if $\ker f_*$ is finitely generated.

Definition

An epimorphism $\phi\colon G \to \mathbb{Z}$ is an *algebraic fibring* if and only if $\ker \phi$ is finitely generated.

How do free-by-cyclic groups fibre?

Theorem (K. 2018)

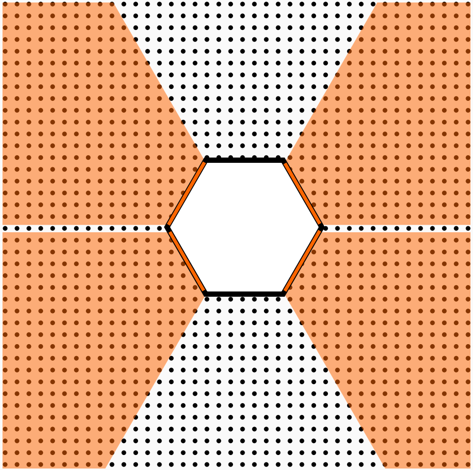
Let $G = F_n \rtimes \mathbb{Z}$. There exists a polytope P_G controlling which $\phi\colon G \to \mathbb{Z}$ algebraically fibres.

How do free-by-cyclic groups fibre?

Theorem (K. 2018)

Let $G = F_n \rtimes \mathbb{Z}$. There exists a polytope P_G controlling which $\phi\colon G \to \mathbb{Z}$ algebraically fibres.

- Every dot is a map $G \to \mathbb{Z}$
- A dot is a fibring $\Leftrightarrow$ it lies in the orange field
- The orange field is the cone over some faces of P_G



Algorithms

Theorems

Theorem

Let M be a 3-manifold. There exists a polytope P_M controlling which $f\colon M \to \mathbb{S}^1$ fibres.

Theorem

Let $G = F_n \rtimes \mathbb{Z}$. There exists a polytope P_G controlling which $\phi\colon G \to \mathbb{Z}$ algebraically fibres.

Theorems

Theorem

Let M be a 3-manifold. There exists a polytope P_M controlling which $f\colon M \to \mathbb{S}^1$ fibres.

Theorem

Let $G = F_n \rtimes \mathbb{Z}$. There exists a polytope P_G controlling which $\phi\colon G \to \mathbb{Z}$ algebraically fibres.

Theorem

The polytope can be computed algorithmically.

Theorems

Theorem

Let M be a 3-manifold. There exists a polytope P_M controlling which $f\colon M \to \mathbb{S}^1$ fibres.

Theorem

Let $G = F_n \rtimes \mathbb{Z}$. There exists a polytope P_G controlling which $\phi\colon G \to \mathbb{Z}$ algebraically fibres.

Theorem

The polytope can be computed algorithmically.

Theorem

And so can the orange marking.

Theorems

Theorem

Let M be a 3-manifold. There exists a polytope P_M controlling which $f\colon M \to \mathbb{S}^1$ fibres.

Theorem

Let $G = F_n \rtimes \mathbb{Z}$. There exists a polytope P_G controlling which $\phi\colon G \to \mathbb{Z}$ algebraically fibres.

Theorem

The polytope can be computed algorithmically.

Theorem (Gardam–K. 2020)

The polytope can be computed algorithmically.

Theorem

And so can the orange marking.

Theorems

Theorem

Let M be a 3-manifold. There exists a polytope P_M controlling which $f\colon M \to \mathbb{S}^1$ fibres.

Theorem

Let $G = F_n \rtimes \mathbb{Z}$. There exists a polytope P_G controlling which $\phi\colon G \to \mathbb{Z}$ algebraically fibres.

Theorem

The polytope can be computed algorithmically.

Theorem (Gardam–K. 2020)

The polytope can be computed algorithmically.

Theorem

And so can the orange marking.

Theorem (Gardam–K. 2020)

And so can the orange marking (modulo a conjecture).

The fibred faces

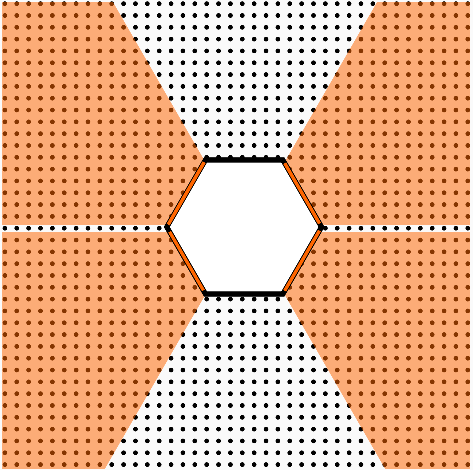
The aim

Theorem

Let $G = F_n \rtimes \mathbb{Z}$. There exists a polytope P_G controlling which $\phi\colon G \to \mathbb{Z}$ algebraically fibres.

Theorem (Gardam–K. 2020)

The orange marking can be effectively computed, modulo a conjecture.



Thurston norm

Back to 3-manifolds: The Thurston poytope P_M is the unit ball of the *Thurston norm*

$$\| \cdot \|_T \colon H^1(M;\mathbb{R}) \to [0,\infty)$$

Definition (Thurston norm)

To every coclass $\phi \in H^1(M;\mathbb{R})$ Poincaré duality associated a dual class in $H_2(M;\mathbb{R})$. Such a class can be represented by an embedded surface Σ. The Thurston norm is (roughly)

$$\|\phi\|_T = \min_{\Sigma} \left(-\chi(\Sigma) \right)$$

When ϕ is fibred with kernel $\pi_1(\Sigma)$, then $\|\phi\|_T = -\chi(\Sigma) = -\chi(\ker \phi)$.

L^2 perspective

Theorem (Friedl–Lück)

When M is virtually fibred, then for every primitive $\phi \in H^1(M;\mathbb{Z})$

$$\|\phi\|_T = -\chi^{(2)}(\ker \phi)$$

L^2 perspective

Theorem (Friedl–Lück)

When M is virtually fibred, then for every primitive $\phi \in H^1(M;\mathbb{Z})$

$$\|\phi\|_T = -\chi^{(2)}(\ker \phi)$$

Theorem (Friedl–Lück; Funke–K.)

When $G = F_n \rtimes \mathbb{Z}$, then the map

$$\phi \mapsto -\chi^{(2)}(\ker \phi)$$

for an epimorphism $\phi\colon G \to \mathbb{Z}$ extends to a semi-norm $H^1(G;\mathbb{R}) \to [0,\infty)$, and its unit ball is P_G.

The conjecture

Theorem

For virtually fibred 3-manifolds, $\|\phi\|_T = -\chi^{(2)}(\ker \phi)$ and the unit ball of the norm is P_M.

Theorem

For free-by-cyclic groups, $\phi \mapsto -\chi^{(2)}(\ker \phi)$ is a semi-norm with unit ball P_G.

Meta-theorem

$\|\phi\|_T$ tells us about the smallest way of representing ϕ.

The conjecture

Theorem

For virtually fibred 3-manifolds, $\|\phi\|_T = -\chi^{(2)}(\ker\phi)$ *and the unit ball of the norm is* P_M*.*

Theorem

For free-by-cyclic groups, $\phi \mapsto -\chi^{(2)}(\ker\phi)$ *is a semi-norm with unit ball* P_G*.*

Meta-theorem

$\|\phi\|_T$ tells us about the smallest way of representing ϕ.

Conjecture (Gardam–K. 2020)

For en epimorphism $\phi\colon G \to \mathbb{Z}$*, we have* $-\chi^{(2)}(\ker\phi)$ *equal to* $\min\{-\chi(A)\}$ *where G can be written an HNN extension inducing* ϕ *with base group A.*

... is sometimes a theorem

Conjecture (Gardam–K. 2020)

For en epimorphism $\phi\colon G \to \mathbb{Z}$, *we have* $-\chi^{(2)}(\ker\phi)$ *equal to* $\min\{-\chi(A)\}$ *where G can be written an HNN extension inducing* ϕ *with base group A.*

Theorem (Henneke–K.)

The conjecture is true when the free-by-cyclic group G is a one-relator group.

Thank you!