Parabolic Subgroups of Infinite Type Artin Groups

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Background on Artin Groups

Artin Groups

Let Γ be a finite simple graph with vertices $S = \{s_1, s_2, \dots s_n\}$ and edges labeled with integers $m_{i,j} \geq 2$.

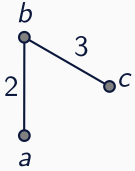
Definition

The *Artin group*, A_Γ, associated to Γ is the group with presentation

$$A_\Gamma = \langle s_1, \dots, s_n \mid \underbrace{s_i s_j s_i \dots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \dots}_{m_{ij} \text{ terms}} \text{ for all } i \neq j \rangle .$$

Examples

Γ :



$$A_\Gamma = \langle a, b, c \mid ab = ba, bcb = cbc \rangle$$

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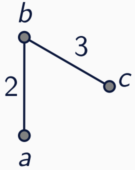


$$A_\Gamma = \langle a, b, c \mid ab = ba, bcb = cbc \rangle$$

Other examples include:

- Free groups
- Free abelian groups and other right-angled Artin groups
- Braid groups
- Free products and direct products of other Artin groups

Definition

The *Coxeter group*, W_Γ, associated to Γ is the group with presentation

$$W_\Gamma = \langle s_1, \dots, s_n \mid s_i^2 = 1, \underbrace{s_i s_j s_i \dots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \dots}_{m_{ij} \text{ terms}} \text{ for all } i \neq j \rangle .$$

Inspiration- Reflections

Examples of Coxeter groups

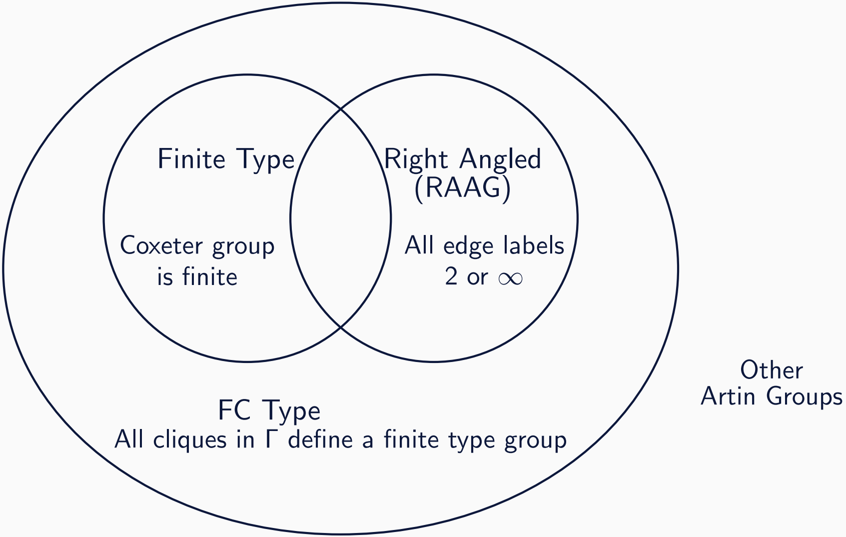
- Symmetric groups
- Dihedral groups
- Weyl groups

Each Coxeter group can be constructed as a group generated by reflections in a vector space.

Finite Type

Artin groups whose corresponding Coxeter group is finite are called Artin groups of *finite type*.

Types of Artin groups



Parabolic subgroups

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Theorem (Van der Lek 1983)

$$gA_Tg^{-1} \cong A_{\Gamma_T}$$

Where Γ_T is the subgraph of Γ spanned by the vertices in T.

Open Questions

Avenues for investigation

- Describe the centrilizers or normalizers for given parabolic subgroup
- Given two parabolic subgroups P, Q when is $P \cap Q$ also parabolic.

A Challenge: Ribbon elements (Godelle 2003)

Example:

Γ :

There exists an element $r \in A_\Gamma$ such that $rsr^{-1} = t$.

Some results

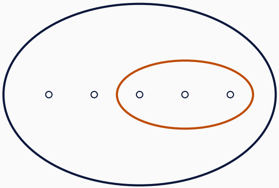
Theorem (MW 2019, Cumplido et al 2019)

Suppose that A_Γ is a FC type Artin group, and P, Q are finite type parabolic subgroups. Then $P \cap Q$ is also a finite type parabolic subgroup.

Analog to the Curve complex

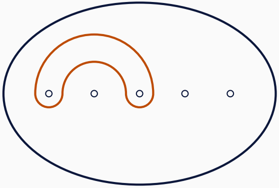
There is a bijection

Curves in punctured disc $\longleftrightarrow$ Parabolic subgroups in braid group



$$A_{\{s_3,s_4\}}$$

Subgroup generated by s_3 and s_4



$$s_2 A_{\{s_1\}} s_2^{-1}$$

Conjugate of subgroup generated by s_1

Definition of Complex of Parabolic Subgroups

Definition (Morris-Wright 2019, Cumplido et al 2019)

Given an Artin group of FC type we define the *complex of parabolic subgroups*

Vertices are proper finite type irreducible parabolic subgroups

An edge appears between P and Q if $z_P z_Q = z_Q z_P$ where z_P is the element that generates the center of P.

Unification of known complexes

The complex of parabolic subgroups

- Agrees with curve complex in braid group case. (Cumplido et al, 2019)
- Agrees with the extension graph (Kim and Koberda, 2013) in the right-angled case. (MW 2019)

Other Known Properties

- Each n-simplex represents a $\mathbb{Z}^n$ subgroup.
- (Almost always) connected and infinite diameter (Kim & Koberda 2013, Calvez & Weist 2019, MW 2019)

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NEXT QUESTION: Is this complex δ-hyperbolic?

Geometric tools

Deligne Complex: Brief summary

A CAT(0) cube complex for FC type Artin groups.

Vertices are cosets gA_T where A_T is finite type

For any pair $gA_T \subset hA_{T'}$ spans a cube of dimension $|T' \setminus T|$.

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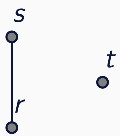
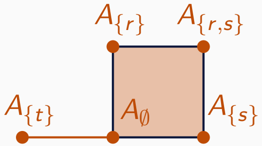
Stabilizer of the vertex gA_T is the parabolic subgroup $gA_T g^{-1}$

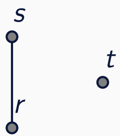
Example

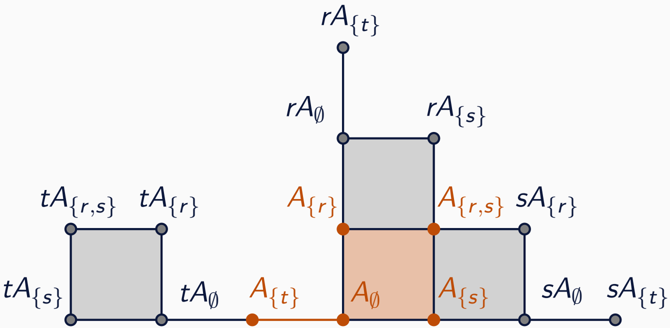




A subcomplex of the Deligne complex for the shown graph with fundamental domain in orange.

Example





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Goal

Theorem (MW 2019, Cumplido et al 2019)

Suppose that A_Γ is a FC type Artin group, and P, Q are finite type parabolic subgroups. Then $P \cap Q$ is also a finite type parabolic subgroup.

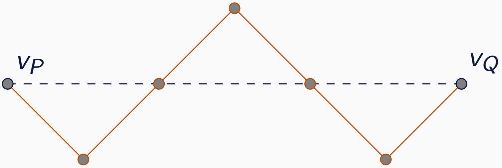
Let P, Q be finite type parabolic subgroups. Choose v_P and v_Q vertices in the Deligne complex fixed by P and Q respectively.

Proof

There is a unique geodesic between v_P and v_Q.

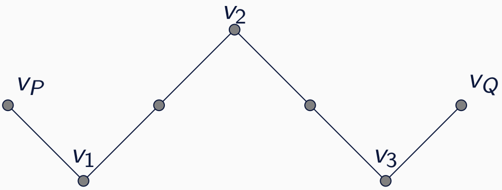
This geodesic is fixed by $P \cap Q$

Proof



$P \cap Q$ fixes a shortest edge path between v_P and v_Q.

Proof



Inclusion of subgroups determines directions on these edges.

This determines a sequence of *turning points*

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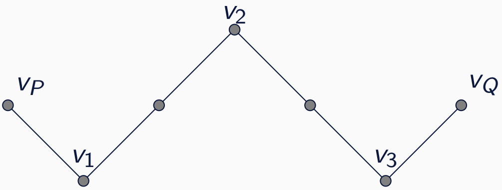

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In this example:

$Stab(v_1) \subset Stab(v_2) \cap Stab(v_P)$

$Stab(v_3) \subset Stab(v_2) \cap Stab(v_Q)$

Proceed by induction

Proceed by induction on the number of turning points.

Let $R = Stab(v_{n-1})$.

Assume $P \cap R$ is a finite type parabolic subgroup.

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Two cases

Case 1 $R \subset Q$

$P \cap Q$ fixes the entire edge path, so $P \cap Q \subset R$.

Together this implies $P \cap Q = P \cap R$.

Case 2 $Q \subset R$

In this case, P, Q both subgroups of the finite type group $P \cap R$ and so we reduce to the finite type case shown by Cumplido et al,

Thank you!

Parabolic subgroups of Artin groups of FC type, *Journal of Pure and Applied Algebra*, to appear

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