Action of the Cremona group on a CAT(0) cube complex

Virtual Geometric Group Theory conference

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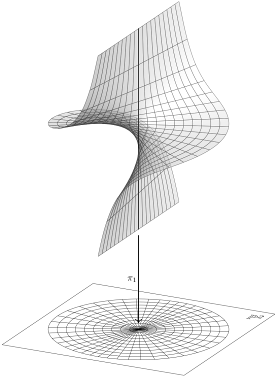
joint work with Christian Urech

Birational geometry

- A **birational transformation** between two surfaces is an isomorphism between two dense open subsets.



- *Example:* **Blow-up** of a point $\pi_1 : S \to S'$.
  - π_1 induces an isomorphism between $S \setminus E_p$ and $S' \setminus p$.
  - $E_p \simeq \mathbb{P}^1$ is called **exceptional divisor**.

- Given S, the set of birational transformations from S to S with the composition form a group, $\mathrm{Bir}(S)$.

Cremona group of rank 2

The **Cremona group** of rank 2, denoted by $\mathrm{Bir}(\mathbb{P}^2_k)$, is the group of birational transformations of $\mathbb{P}^2_k$.

- L. Cremona introduced this group in 1863-1865.
- Various aspects of this group:
  - algebraical,
  - dynamical,
  - topological,
  - geometrical...

Aim of this talk

Construct an action of the Cremona group on a CAT(0) cube complex.

- $\rightsquigarrow$ Gives a new geometric space for the Cremona group of rank 2.
- $\rightsquigarrow$ It has been a step towards the construction of a geometric space for Cremona groups of higher ranks.

- A Cremona transformation f has the following form:

$$\begin{array}{rccc} f: & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \\ & [x:y:z] & \mapsto\dashrightarrow & [f_0(x,y,z):f_1(x,y,z):f_2(x,y,z)] \end{array}$$

  where $f_0, f_1, f_2 \in k[x, y, z]$ are homogeneous polynomials of the same degree without common factor.

- $\deg f := \deg f_i$

- $\cap\{f_i = 0\}$ set of points not well-defined.

Cremona group of rank 2

- $f : [x : y : z] \dashrightarrow [f_0(x,y,z) : f_1(x,y,z) : f_2(x,y,z)]$
- $\deg f := \deg f_i$
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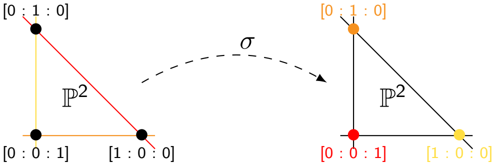
Examples

$*$ $\mathrm{Aut}(\mathbb{P}^2) = \{f \in \mathrm{Bir}(\mathbb{P}^2) \mid \deg f = 1\} \simeq \mathrm{PGL}(3,k)$.

$\{[x : y : z] \dashrightarrow [ax + by + cz : dx + ey + fz : gx + hy + iz]\}$

$*$ $\sigma : [x : y : z] \dashrightarrow [yz : xz : xy]$.

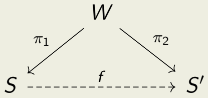


Base-points

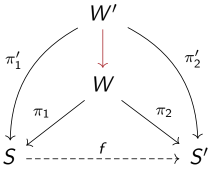
Zariski theorem

Theorem

Let $f : S \dashrightarrow S'$ be a birational transformation between surfaces. Then there exists a surface W and compositions of blow-ups $\pi_1 : W \to S$, $\pi_2 : W \to S'$ such that:

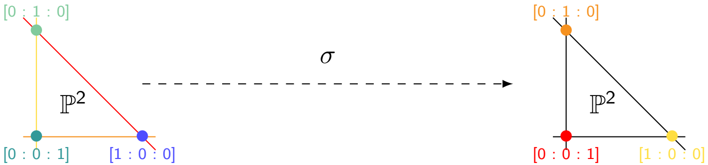


- <u>Remark:</u> W can be chosen minimal and we call it **minimal resolution** of f.

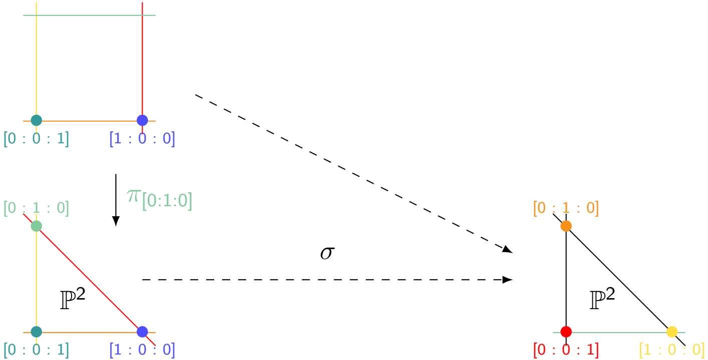


Base-points

Example



Example


[0 : 0 : 1]
[1 : 0 : 0]
$\pi_{[0:1:0]}$
[0 : 1 : 0]
$\mathbb{P}^2$
[0 : 0 : 1]
[1 : 0 : 0]
σ
[0 : 1 : 0]
$\mathbb{P}^2$
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Base-points

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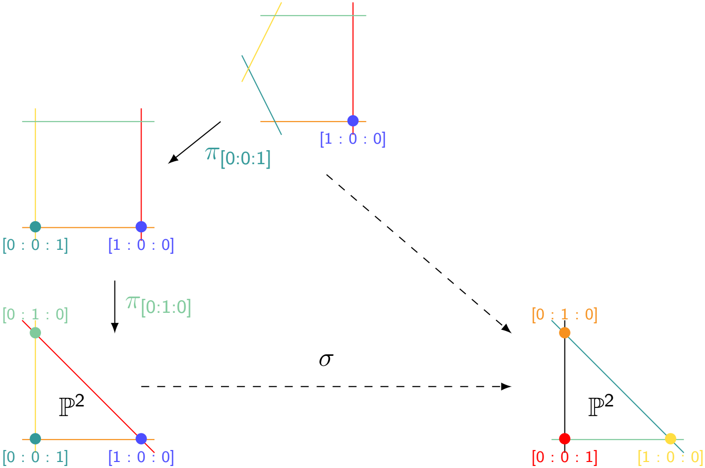

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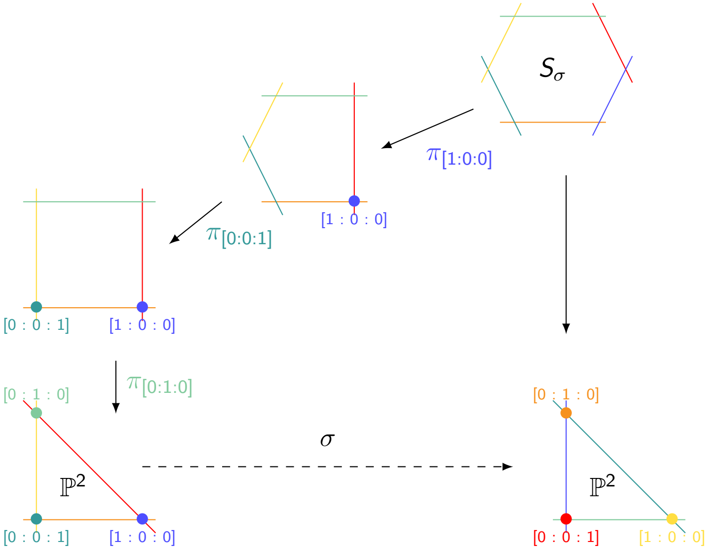

S_σ
$\pi_{[1:0:0]}$
[1 : 0 : 0]
$\pi_{[0:0:1]}$
[0 : 0 : 1]
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Base-points

Definition

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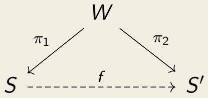


- The points blown-up by π_1 in the minimal resolution of f are called **base-points** of f, denoted by $\mathcal{B}(f)$.
- Remark: They do not all lie in S. For instance $[x : y : z] \mapsto [xz : x^2 - yz : z^2]$.

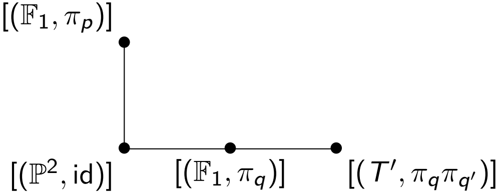
The blow-up complex

Construction

- **Vertices:** $[(S, \varphi)]$
  - $*$ S birational surface,
  - $*$ $\varphi : S \dashrightarrow \mathbb{P}^2$ birational map,
  - $*$ $(S, \varphi) \sim (S', \varphi')$ iff $\varphi'^{-1}\varphi : S \xrightarrow{\sim} S'$ is an isomorphism.

- **Edges:** $[(S, \varphi)] \bullet\!\!-\!\!-\!\!\bullet [(T, \psi)]$
  if $\psi^{-1}\varphi$ is a blow-up or the inverse of a blow-up.

Example ($p, q \in \mathbb{P}^2$, $q' \in E_q$).



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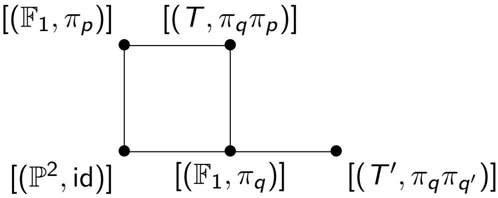

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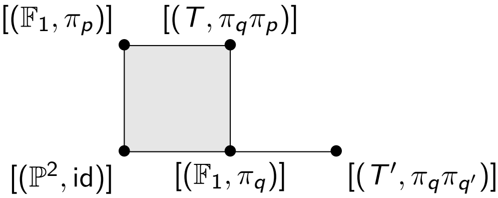

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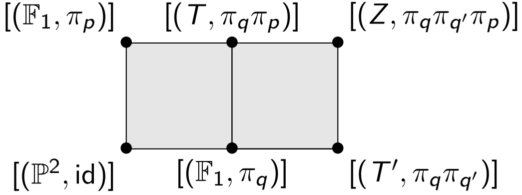
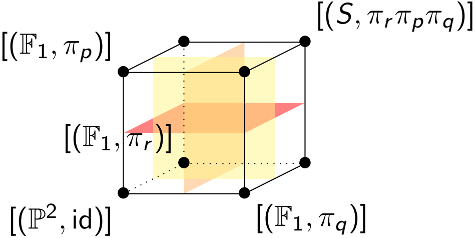


Construction

- ***n*-cubes:** $[(S_1, \varphi_1)], \ldots, [(S_{2^n}, \varphi_{2^n})]$, if there exists $1 \leqslant r \leqslant 2^n$ such that for any $1 \leqslant j \leqslant 2^n$:
  - $p_1, \ldots, p_n \in S_r$ distinct,
  - $\varphi_r^{-1}\varphi_j : S_j \to S_r$ is the blow-up of $E \subset \{p_1, \ldots, p_n\}$.

Example ($p, q, r \in \mathbb{P}^2$).



The blow-up complex

Remarks

The blow-up complex is:

- not locally compact,
- infinite dimensional,
- oriented: from $[(S, \varphi)]$ to $[(S', \varphi')]$ if $\varphi'^{-1}\varphi$ is the blow-up of a point of S'.

Theorem (2020; A. L. and Christian Urech)

The blow-up complex is a CAT(0) cube complex.

Theorem (2020; A. L. and Christian Urech)

The blow-up complex is a CAT(0) cube complex.

"Proof".

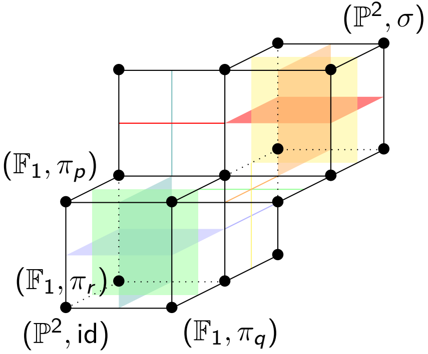
- connected: Zariski theorem.
- simply connected: Let $v_1, \ldots, v_n$ vertices of a loop.
  * Choose one of minimal height ρ: v_{i_0}. Then $\rho(v_{i_0-1}) = \rho(v_{i_0+1}) = \rho(v_{i_0}) + 1$.
  * v_{i_0-1}, v_{i_0}, v_{i_0+1} and v'_{i_0} form a square, so replace v_{i_0} by v'_{i_0}.
  * Zariski theorem: existence of a minimal surface dominating fixed representatives of the vertices $v_1, \ldots, v_n$. It dominates also a representative of v'_{i_0}.
- links are flag.

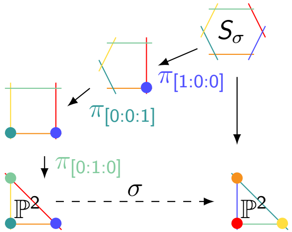
□

Action of the Cremona group on the blow-up complex

Let $f \in \mathrm{Bir}(\mathbb{P}^2)$ and $[(S,\varphi)]$ be a vertex,

$$f \bullet [(S,\varphi)] = [(S, f\varphi)].$$





Some Results

- Nice correspondence: for $f \in \mathrm{Bir}(\mathbb{P}^2)$,
  - $\mathrm{dist}\left([(\mathbb{P}^2, \mathrm{id})], [(\mathbb{P}^2, f)]\right) = 2|\mathcal{B}(f)|$,
  - $\ell(f) = 2 \lim_{n\to\infty} \frac{|f^n|}{n}$,
  - elliptic elements are elements conjugated to an automorphism of a surface (called **regularizable**)

Theorem ('01; J. Diller - C. Favre / '20; L. - C. Urech)

Every birational transformation is conjugate to an algebraically stable model.

- Remark: We prove it over any field.

Some Results

Proposition (2020; L. - C. Urech)

Let $G \subset \mathrm{Bir}(\mathbb{P}^2)$ such that

- G has property FW, or
- there exists $K \geqslant 0$ such that for any $g \in G$

$$\deg(g) \leqslant K,$$

then G is regularizable.

- Remarks:
  - ∗ The first result has been done by Cantat-Cornulier over algebraically closed fields.
  - ∗ The second one is a consequence of Weil theorem.
  - ∗ Our proofs are straightforwards.

Question

Consider a subgroup G of the Cremona group such that each of its elements is regularizable. Does it imply that G is regularizable ?

Thank you!