Quasi-parabolic structures on groups

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[1] Partly joint work with D.Osin, C.Abbott and A.Rasmussen

INTRODUCTION

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$G \curvearrowright \Gamma(G, X)$ is isometric and cobounded

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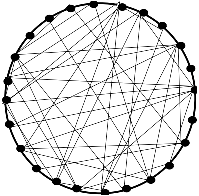

DEFINITION (COMPARING GENERATING SETS; ABO)

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- If $X \subset Y$, then $Y \preceq X$ (Inclusion reversing)

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- If G has a finite generating set X, then $[X]$ is the largest structure
- If $[X] = [Y]$, then $\Gamma(G, X)$ is *quasi-isometric* to $\Gamma(G, Y)$

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Elements of $\mathcal{H}(G)$

$$\updownarrow$$

Equivalence classes of cobounded actions of G on hyperbolic spaces (up to a natural equivalence)

Theorem (ABO)

For any group G,

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- $\mathcal{H}_{gt}(G)$ contains general type structures. i.e. $|\Lambda(G)| = \infty$ and G has no fixed points on ∂X.
- Parabolic actions are never cobounded
- $\mathcal{H}(G)$ is a way to study all possible cobounded actions of a group on hyperbolic spaces, upto q.i.

THEOREM (ABO)

For every $n \in \mathbb{N}$, there exists a group G_n such that

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For every $n \in \mathbb{N}$, there exists a group H_n such that

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Theorem (ABO)

$\mathcal{H}_{qp}(\mathbb{Z} \operatorname{wr} \mathbb{Z})$ *contains an antichain of cardinality continuum.*

- Obtained by factoring through $\mathbb{Z}_n \operatorname{wr} \mathbb{Z}$ acting on the Bass-Serre tree.

QUESTIONS

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4. If $|\mathcal{H}_{qp}(G)| \neq 0$, is $|\mathcal{H}_{\ell}(G)| \leq |\mathcal{H}_{qp}(G)|$?

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There exists a group G such that $|\mathcal{H}_\ell(G)| > |\mathcal{H}_{qp}(G)| > 0$.

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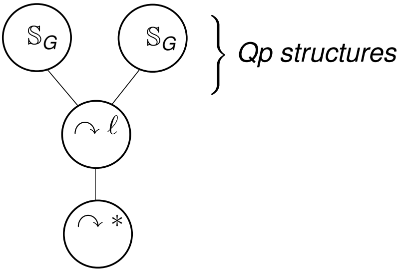

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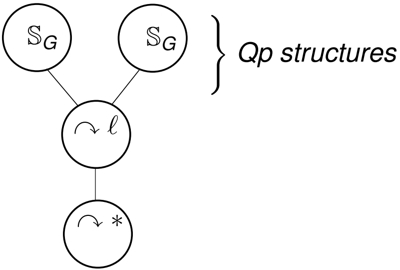

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(2) *If* $G = \mathbb{Z}_n$, *then* $\mathcal{B}(G) = \mathcal{H}(\mathbb{Z}_n \,\mathrm{wr}\, \mathbb{Z})$.

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- Regularity $\Rightarrow$ Common lineal structure
- When $G = \mathbb{Z}_n$, the inclusion is a surjection (Not true in general)

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  Then $|\mathcal{H}_{qp}(K)| = 2$, and $|\mathcal{H}_\ell(K)| = c$.

FURTHER WORK

THEOREM (AR)

Let $G = BS(1,n)$, $n \geq 2$. *Then* $G = \mathbb{Z}\left[\frac{1}{n}\right] \rtimes \mathbb{Z}$ *and* $\mathcal{H}(G)$ *has the following structure.*

FURTHER WORK

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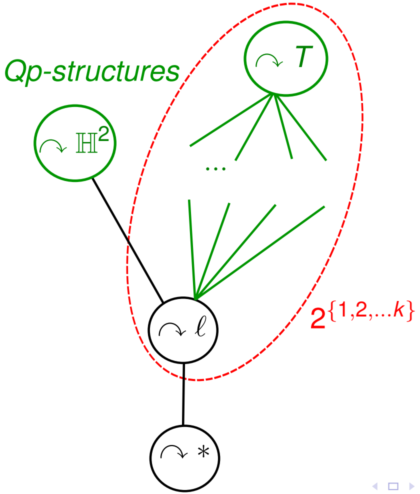

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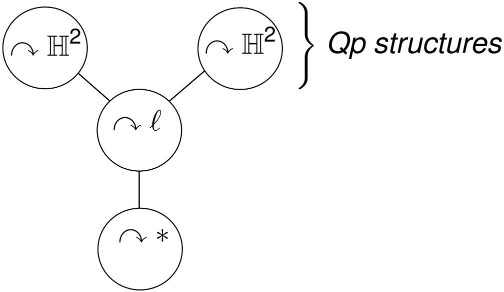

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- Classifying structures on $\mathbb{Z}^n \rtimes_\phi \mathbb{Z}$, where $\phi \in SL_n(\mathbb{Z})$ for $n \geq 3$
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