Beyond Endoscopy

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Langlands' idea

Quasisplit connected reductive groups G_1, G_2 over a global field k.

$\rho : {}^L G_1 \to {}^L G_2$

Should correspond (roughly) to a map

$\rho_* : \{\text{Stable } L\text{-packets for } G_1\} \rightsquigarrow \{\text{Stable } L\text{-packets for } G_2\}$,
locally and globally.

Think of L-packets in terms of stable characters:

For a representation π of $G = G(F)$ (local field) or $G(\mathbb{A}_k)$,
$\Theta_\pi = \operatorname{tr} \pi(\bullet) : \mathcal{S}(G) \to \mathbb{C}$.

For a local L-packet Π, $\Theta_\Pi = \sum_{\pi \in \Pi} \langle \eta(\pi), 1 \rangle \Theta_\pi$.

For a global L-packet Π, $\Theta_\Pi = \prod_v \Theta_{\Pi_v}$,
and we have the stable trace formula (STF) which, very roughly, is a stably invariant distribution $\mathcal{S}(G(\mathbb{A}_k))_G \to \mathbb{C}$ that decomposes as

$$\int_{\psi:\text{global Arthur parameters}} \Theta_\psi d\psi. \tag{1}$$

Langlands' idea

$$\rho : {}^L G_1 \to {}^L G_2$$

Langlands (2000) suggested that there should be a way to compare STF_{G_1} with STF_{G_2}; what does "compare" mean, since they are functionals on different spaces $\mathcal{S}(G_1)_{G_1}, \mathcal{S}(G_2)_{G_2}$?

- There should be a local "transfer operator" $\mathcal{T}_\rho : \mathcal{S}(G_2)_{G_2} \to \mathcal{S}(G_1)_{G_1}$ realizing the local transfer of stable characters:
$$\mathcal{T}_\rho^* \Theta_{\Pi_1} = \Theta_{\rho_* \Pi_1}.$$
- One should be able to extract (through poles of L-functions) the part of STF_{G_2} related to the functorial image of G_1
$\rightsquigarrow$ get a new distribution $\mathrm{STF}^\rho_{G_2} : \mathcal{S}(G_2(\mathbb{A}_k)) \to \mathbb{C}$ such that
$$\mathcal{T}_\rho^* \mathrm{STF}_{G_1} = \mathrm{STF}^\rho_{G_2}.$$
(Caution: oversimplifying!)

An example

E/k quadratic, $T = U_1$, $G = \mathbb{G}_m \times \mathrm{SL}_2$,

${}^LT \xrightarrow{\rho} {}^LG = \mathbb{G}_m \times \mathrm{PGL}_2 = \mathrm{GL}_2 / \mu_2 \xrightarrow{r=\mathrm{Sym}^2} \mathrm{GL}_3$.

Here, r appears because a general vector under the Sym^2 representation is fixed by the image O_2/μ_2 of ${}^LT$. Hence, automorphic representations π in the image of ρ_* will have:

- a pole for $L(\pi, r, s)$ at $s = 1$;
- restriction $\eta_{E/k}$ to the $\mathbb{G}_m$-factor.

We expect to be able to compare the trace formula (Poisson sum) for T with the "r-trace formula" of G (s. Arthur, "Problems beyond endoscopy", and recent papers of Tian An Wong), where (1) is modified by the L-factors

$$\int_{\psi:\text{global Arthur parameters}} L(r \circ \psi, s)\Theta_\psi d\psi,$$

by inserting appropriate non-standard test functions, and one studies the residue (& lower Laurent coefficients) at $s = 1$.

Local transfer

The problem of non-tempered representations: One basic issue that has received a lot of attention is that this transfer is not supposed to behave well for nontempered (non-Ramanujan type) representations.

Locally, there is an easy resolution: require that $\mathcal{T}_\rho^* \Theta_{\Pi_1} = \Theta_{\Pi_2}$ only for tempered L-packets. Work in this direction:

- Langlands, "Singularités et transfert." Studies the lift from $T = U_1$ to $G = \mathrm{SL}_2$ as in the previous slide. Recovers, from character formulas of Harish-Chandra, Sally & Shalika, the formula of Gelfand–Graev–PS for the stable transfer of characters from T to G, and shows that it is adjoint to a transfer operator $\mathcal{T}_\rho : \mathcal{S}(\mathrm{SL}_2)_{\mathrm{SL}_2} \to \mathcal{S}(T)^{\mathbb{Z}/2}$. Here, $\mathbb{Z}/2$ acts by inversion, and the elements in the target can be written as measures on the quotient $T /\!\!/ \mathbb{Z}/2 = \mathbb{A}^1$. Similarly, elements of $\mathcal{S}(\mathrm{SL}_2)_{\mathrm{SL}_2}$ are measures in the trace variable, and the formula reads
$$\mathcal{T}_\rho f(t) = \frac{2}{\mathrm{Vol}(T)\sqrt{|t^2-4|}} \left(\frac{\eta}{|\bullet|} \star_+ f \right)(t) \text{ (additive convolution)}$$

Local transfer

- Daniel Johnstone, generalizes the GGPS formula to GL_n, based on character formulas of Adler–DeBacker–Spice.
- D. Johnstone–Zhilin Luo: describe the kernel of an operator $\mathcal{T}^* : \mathcal{S}(\mathrm{GL}_2)^*_{\mathrm{GL}_2} \to \mathcal{S}(\mathrm{GL}_{n+1})^*_{\mathrm{GL}_{n+1}}$ with the property that $\mathcal{T}^*\Theta_\pi = \Theta_{(\mathrm{Sym}^n)_*\pi}$ for every irreducible representation of $\mathrm{GL}_2(F)$ that is not the twist of Steinberg.
  Natural questions: (1) How to correct this operator to work for Steinberg, as well? (2) Is this the adjoint of a $\mathcal{T} : \mathcal{S}(\mathrm{GL}_{n+1})_{\mathrm{GL}_{n+1}} \to \mathcal{S}(\mathrm{GL}_2)_{\mathrm{GL}_2}$?

Basic problem: Understand local transfer operators for any $\rho : {}^LG_1 \to {}^LG_2$.

Caution: "for any" is very broad, even in the case $G_1 = 1$, where ρ a single Galois/Weil representation.

Global analysis

Globally, we need to isolate the representations not of Ramanujan type from the STF.

Geometric side of the stable trace formula:

$$\sum_{\xi \in \frac{G}{G}(k)} J_\xi(f)$$

(at least, the elliptic part), where $\Xi := \frac{G}{G}$ is the Chevalley quotient $\operatorname{Spec} k[G]^G$ (the "Steinberg–Hitchin base"). If G is simply connected, this is an affine space, and one can imagine a Poisson summation on this space. (Why, though?)

- Frenkel–Langlands–Ngô: The terms $J_\xi(f) = \mathrm{Vol}([G_\xi])O_\xi(f)$ (for regular elliptic classes) can be written as limits of Euler products, $J_\xi(f) = \lim_{s\to 1^+} \prod_v \theta_v(f_v, \xi_v, s) \rightsquigarrow$ try to make sense of 0-th Fourier coefficient in ξ-variable, $\hat{J}_0(f)$ & identify it with $\mathrm{tr}\,\pi(f)$ for $\pi =$ the trivial representation.

- Altuğ: Poisson summation formula on $\Xi = \frac{G}{G}$ for $G = \mathrm{GL}_2$, $r = \mathrm{Std}$ (let's think of PGL_2, for simplicity, so $\Xi = \mathbb{A}^1$), using the approximate functional equation to handle the L-factors $\mathrm{Vol}([G_\xi])$, and a detailed study of the various terms of the trace formula.
  An interesting feature: after Fourier transform, Kloosterman sums appear.
- Question of Arthur: For $G = \mathrm{GL}_n$, we have a decomposition of the discrete automorphic spectrum into Arthur parameters (Mœglin–Waldspurger decomposition):
  $$\hat{G}^{\mathrm{Aut}}_{\mathrm{disc}} = \prod_{m|n} G^{\mathrm{Aut}}_{mn},$$
  corresponding to irreducible Arthur parameters $\mathcal{L}_k \times \mathrm{SL}_2^A \to \mathrm{GL}_n$ of the form $\varphi \otimes \mathrm{Sym}^d$, with $d = \frac{n}{m} - 1$.
  Is there a decomposition $\Xi^* = \sqcup_{m|n} \Xi_m$ of the dual of the Hitchin base (mod center), so that after Fourier transform the spectral and geometric terms of the decompositions above match?

The role of the Kuznetsov formula

Early on, Sarnak pointed out that the *Kuznetsov formula* already comes with the Ramanujan-type representations isolated, hence might be more suitable for Beyond Endoscopy.

- Realized in the thesis of Akshay Venkatesh for $r = \mathrm{Sym}^1$ and $r = \mathrm{Sym}^2$ ($G = \mathrm{GL}_2$). (Will return.)
- Several papers of Eddie Herman: studied the r-Kuznetsov formula for $r = \mathrm{Std}$ of GL_2, $\otimes$ of $\mathrm{GL}_2 \times \mathrm{GL}_2$ (Rankin–Selberg), tensor square (Asai) for $\mathrm{GL}_2(E)$, $(E : k) = 2$, and proved several results, including:
  - relatively self-contained, trace-formula-theoretic proofs of the functional equations of $L(\pi, \mathrm{Std}, s)$ and $L(\pi, \mathrm{Asai}, s)$;
  - a calculation of the residue of the r-trace formula for $r = \otimes$, in terms of representations of the form $\pi \otimes \tilde{\pi}$.
- As I will explain, all previous efforts based on the Selberg trace formula are actually related to the Kuznetsov formula (joint work w. Chen Wan).

But first, we need to introduce a more general kind of comparison.

The relative trace formula of Jacquet

Given a (quasiaffine) spherical G-variety X, the relative trace formula for $X \times X/G$ (diagonal action of G): a global analog of the local Plancherel formula:

The *local* Plancherel formula is a spectral decomposition for the pairing

$$\mathcal{S}(X(F)) \otimes \mathcal{S}(X(F)) \ni \Phi_1 \otimes \Phi_2 \mapsto \langle \Phi_1, \Phi_2 \rangle_X = \int_X \Phi_1(x)\Phi_2(x)dx.$$

(Here: $X = X(\mathbb{Q}_p)$, $p \leq \infty$.)

The *global* RTF is a spectral decomposition for the pairing

$$\mathcal{S}(X(\mathbb{A}_k)) \otimes \mathcal{S}(X(\mathbb{A}_k)) \ni \Phi_1 \otimes \Phi_2 \mapsto \mathrm{RTF}(\Phi_1 \otimes \Phi_2) :=$$

$$\int_{[G]} \sum \Phi_1(g) \cdot \sum \Phi_2(g),$$

where $\sum \Phi_i(g) = \sum_{\gamma \in X_i(\mathbb{Q})} \Phi_i(\gamma g)$, the theta series of Φ_i, is an automorphic function.

The relative trace formula of Jacquet

Another way to think of the RTF is as a distribution of the points of a quotient stack $\mathfrak{X} = X \times X/G$:

$$\mathrm{RTF} = \sum_{\xi \in \mathfrak{X}} \mathrm{ev}_{\xi} .$$

In particular, if defined correctly,

- does not depend on the presentation of the stack (e.g, if $X = H\backslash G$, can write $\mathfrak{X} = H\backslash G/H$);
- involves “pure inner forms” of X (as in the Gan–Gross–Prasad conjectures).

Arthur–Selberg trace formula: the RTF for $X = H$, $G = H \times H$. (Indeed, trace of an operator = Hilbert–Schmidt inner product of two operators.)

Kuznetsov formula KTF: the RTF for $X = (N, \psi)\backslash G =$ the Whittaker model.

(Will be working with stable versions throughout.)

Relative functoriality

One can associate an L-group ${}^LG_X$ with a map to ${}^LG$.
(Gaitsgory–Nadler, S.–Venkatesh, Knop–Schalke.)

Given spherical pairs (G, X) and (G', Y), and a morphism

$$\rho : {}^LG_X \to {}^LG'_Y,$$

we should have a functorial lift
$\rho_* : \{$packets of X-distinguished representations of $G\} \to$
$\{$ packets of Y-distinguished representations of $G'\}$,
realized by

- locally, a transfer operator $\mathcal{T}_\rho : \mathcal{S}(Y \times Y)_{G'} \to \mathcal{S}(X \times X)_G$, pulling back relative characters $\mathcal{T}_\rho^* \Theta_\Pi^X = \Theta_{\rho_*\Pi}^Y$;
- globally, a way to extract a piece RTF_Y^ρ of RTF_Y, such that $\mathcal{T}_\rho^* \circ \mathrm{RTF}_X = \mathrm{RTF}_Y^\rho$.

"Beyond endoscopy" for the RTF

Easiest (but quite nontrivial!) case: when ${}^LG_X = {}^LG'_Y$; then, $\mathrm{RTF}^{\rho}_Y = \mathrm{RTF}_Y$, and we should have

$$\begin{tikzcd}
\mathcal{S}_{L_X}(\mathfrak{Y}(\mathbb{A}_k)) \arrow[rr, "\mathcal{T}_\rho"] \arrow[dr, "\mathrm{RTF}_Y"'] & & \mathcal{S}(\mathfrak{X}(\mathbb{A}_k)) \arrow[dl, "\mathrm{RTF}_X"] \\
& \mathbb{C} &
\end{tikzcd}.$$

Example–problem: For any quasisplit G, one should be able to compare STF_G with KTF_G.

Important caveat: The spectral side of a general RTF is weighted by L-functions, e.g., for KTF:

$$\int_{\substack{\psi:\text{global Langlands parameters}\\ \text{(Ramanujan type)}}} \frac{1L_X(\psi)}{L(\psi,\mathrm{Ad},1)}\Theta_\psi d\psi.$$

We need to introduce L-functions by using nonstandard test functions, e.g., to compare with STF, introduce $L_X(\psi) = L(\psi, \mathrm{Ad}, 1)$.

Comparisons with the Kuznetsov formula

The base case seems to be the Kuznetsov formula, where there are no L-functions in the numerator.

Conjecture: For every X, there is a comparison between RTF_X and KTF_{G_X}.

Realized in the following cases:

- Rudnick's 1990 thesis: KTF $\leftrightarrow$ STF for holomorphic discrete series of GL_2 (i.e., Petersson's "simple KTF"). Altuğ's work, as I will explain, amounts to the same for the entire Selberg trace formula.
- S.– all (homogeneous, affine) spherical varieties with ${}^LG = \mathrm{SL}_2$ or PGL_2.
- S.–Chen Wan (in progress): KTF $\leftrightarrow$ STF for GL_n.

Only case where the global comparison has been completed with ${}^LG_X \neq {}^LG'_Y$:

- Venkatesh's thesis, $\mathrm{KTF}_T \leftrightsquigarrow \mathrm{KTF}_{\mathrm{SL}_2}$.

Formulas for the transfer operators: is there any structure?

Rank one:

Theorem (S.)
Let X be of rank one:
$\mathrm{GL}_n \backslash \mathrm{PGL}_{n+1}, \mathrm{SO}_{2n} \backslash \mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2 \backslash \mathrm{Sp}_{2n}, \mathrm{Spin}_9 \backslash F_4, \mathrm{SL}_3 \backslash G_2,$
or $\mathrm{SO}_{2n-1} \backslash \mathrm{SO}_{2n}, \mathrm{Spin}_7 \backslash \mathrm{Spin}_8, G_2 \backslash \mathrm{Spin}_7,$
and $\mathfrak{Y} = (N, \psi) \backslash G^* / (N, \psi)$*, the Kuznetsov quotient, where* $G^* = \mathrm{PGL}_2$ *for the first group,* SL_2 *for the second.*

The transfer operator $\mathcal{T} : \mathcal{S}^-_{L_X}(\mathfrak{Y}) \xrightarrow{\sim} \mathcal{S}(\mathfrak{X})$
is given by an explicit composition of Fourier transforms.

For example, for $X = \mathrm{SO}_3 \backslash \mathrm{SO}_4 = \mathrm{SL}_2$, where $X \times X /\!\!/ G = \frac{\mathrm{SL}_2}{\mathrm{SL}_2} \simeq \mathbb{A}^1$ is parametrized by the trace t, and $(N, \psi) \backslash \mathrm{SL}_2 / (N, \psi)$ is represented by elements $\begin{pmatrix} & -\zeta^{-1} \\ \zeta & \end{pmatrix}$ the operator is given by

$$\mathcal{T} f(\zeta) = |\zeta| \widehat{f(\frac{1}{\bullet})}(\zeta).$$

Remarks

- The space $\mathcal{S}^-_{L_X}(\mathfrak{Y})$ is enlarged by the test functions that will produce the desired L-function on the spectral side of the RTF.
- What does it mean that "the transfer operator is given" by this formula?
    - The general theorem is a theorem on *matching*.
    - There are cases where we know a fundamental lemma for the Hecke algebra and statements on transfer of characters: S. ($X = T\backslash \mathrm{PGL}_2, \mathrm{SL}_2$), Wee Teck Gan–Xiaolei Wan ($X = \mathrm{SO}_n\backslash\mathrm{SO}_{n+1}$, variants of the method work more generally).
    - Johnstone–Krishna (had been) working on the fundamental lemma.

The formula for rank one

$X = H\backslash G$ with dual group ${}^L G_X = \mathrm{SL}_2$ or PGL_2.

There an L-value (= graded representation ${}^L G_X \to \mathrm{GL}(V_X)$) associated to X.

If ${}^L G_X = \mathrm{SL}_2$, then $V_X = \mathrm{Std} \oplus \mathrm{Std}$, with gradings depending on X.

If ${}^L G_X = \mathrm{PGL}_2$, then $V_X = \mathrm{Ad}$, again with grading depending on X.

Its RTF is weighted by $L_X(\psi)$ (relative to the Kuznetsov formula).

The miracle of the formula for the transfer operator is that it is determined completely by L_X; for example, if ${}^L G_X = \mathrm{PGL}_2$, $X \times X /\!\!/ G \simeq \mathbb{A}^1$, $(N,\psi)\backslash \mathrm{SL}_2 /(N,\psi)$ is represented by elements $\begin{pmatrix} & -\zeta^{-1} \\ \zeta & \end{pmatrix}$, and the operator is given by

$$\mathcal{T} f(\zeta) = |\zeta| \widehat{|\bullet|^{1-s} f(\frac{1}{\bullet})}(\zeta), \text{ while } L_X = L(\mathrm{Ad}, s).$$

(generalizing the case $X = \mathrm{SL}_2$, where $s = 1$).

Local theory behind Venkatesh's thesis

Theorem (S.)
For ${}^L T \hookrightarrow {}^L G = \mathrm{PGL}_2$, $\eta \leftrightarrow T$ quadratic character, there is a transfer operator

$$\mathcal{T} : \mathcal{S}^-_{L(\mathrm{Sym}^2,1)}(N,\psi\backslash G/N,\psi) \to \mathcal{S}(T)^{\mathbb{Z}/2}$$

given by the formula

$$\mathcal{T} f(a) = \int f(\frac{t(a)}{u})\eta(\frac{t(a)}{u})\psi(u)du,$$

where again we represent Kuznetsov orbital integrals as functions in the variable $\begin{pmatrix} & -\zeta^{-1} \\ \zeta & \end{pmatrix}$*, and* $t(a)$ *is the image of* $a \in T$ *in* $T /\!\!/ (\mathbb{Z}/2) = \mathbb{A}^1$*.*

The transfer operator satisfies the fundamental lemma for the Hecke algebra, and the condition on pullback of characters.

Joint work with Chen Wan

Let G be any reductive group, and $X = H\backslash G$ a *strongly tempered variety*, i.e.: for any tempered representation π of $G(F)$, the integral

$$J_\pi : g \mapsto \int_{H(F)} \Theta_\pi(hg)dh$$

converges as a generalized function of g. (After smoothing, this is the Ichino–Ikeda integral of matrix coefficients.)

Then, J_π represents a relative character on $H\backslash G/H$. Replace Θ_π by its stable character Θ_Π. One hopes that this is dual to a map

$$\mathcal{T} : \mathcal{S}(H\backslash G/H) \to \mathcal{S}(\frac{G}{G}),$$

which is the transfer map $\mathrm{RTF}_X \leftrightarrow \mathrm{STF}_G$. (Notice: under the strongly tempered assumption, ${}^L G_X = {}^L G$.)

We apply this to $X = (N,\psi)\backslash G$, i.e., the Kuznetsov formula, where we can *prove* the existence of $\mathcal{T}$.

Formulas for KTF $\leftrightarrow$ STF

$G = \mathrm{GL}_n$. Represent Kuznetsov orbital integrals as measures in the variable $\begin{pmatrix} & & \zeta_n \\ & \iddots & \\ \zeta_1 & & \end{pmatrix}$, orbital integrals for the adjoint quotient as measures in the coefficients of the characteristic polynomial

$$x^n - t_1 x^{n-1} + t_2 x^{n-2} - \cdots + (-1)^n t_n,$$

or alternatively in the variables
$u_1 = t_1 = \mathrm{tr}\, g, u_2 = \frac{t_2}{t_1} = \frac{\mathrm{tr} \wedge^2 g}{\mathrm{tr}\, g}, \ldots, u_i = \frac{t_i}{t_{i-1}} = \frac{\mathrm{tr} \wedge^i g}{\mathrm{tr} \wedge^{i-1} g}, \ldots$

Then, we have the following formulas for

$$\mathcal{T} : \mathcal{S}(N, \psi \backslash G / N, \psi) \to \mathcal{S}(\frac{G}{G}) :$$

$$n = 2\text{:}\ \mathcal{T} f(u_1, u_2) = \int_F f \begin{pmatrix} & -u_2 \kappa \\ u_1 \kappa^{-1} & \end{pmatrix} \psi(\kappa) d\kappa.$$

(Specializing to SL_2, this is the transfer operator that we saw before.)

$n = 2$:

$$\mathcal{T}f(u_1,u_2) = \int_F f\begin{pmatrix} & -u_2\kappa \\ u_1\kappa^{-1} & \end{pmatrix}\psi(\kappa)d\kappa.$$

$n = 3$:

$$\mathcal{T}f(u_1,u_2,u_3) = \int_{F^3} f\left(\begin{pmatrix} & & \kappa_2\lambda u_3 \\ & -\kappa_1\kappa_2^{-1}u_2 & \\ \kappa_1^{-1}\lambda^{-1}u_1 & & \end{pmatrix}\right)\cdot$$

$$\cdot\psi\left(\kappa_1+\kappa_2+\lambda+\frac{u_3}{u_2}\frac{\kappa_2^2}{\kappa_1}+\frac{u_2}{u_1}\frac{\kappa_1^2}{\kappa_2}\right)\cdot\left|\frac{\lambda^{-1}}{u_1^2u_2}\right|d\kappa_1 d\kappa_2 d\lambda.$$

$n = 4$:

$$\mathcal{T}f(u_1,u_2,u_3,u_4) = \int_{F^6} f(\begin{pmatrix} & & & -u_4\mu\lambda_2\kappa_3 \\ & & u_3\cdot\frac{\lambda_1\kappa_2}{\kappa_3} & \\ & -u_2\cdot\frac{\kappa_1}{\lambda_2\kappa_2} & & \\ \frac{u_1}{\mu\lambda_1\kappa_1} & & & \end{pmatrix})$$

$$\psi(\mu+\lambda_1+\lambda_2+\kappa_1+\kappa_2+\kappa_3+\frac{u_3\lambda_1\kappa_2^2}{u_2\kappa_1\kappa_3}+\frac{u_2\kappa_1^2}{u_1\kappa_2}+\frac{u_2\kappa_1^2\lambda_1}{u_1\lambda_2\kappa_2}+\frac{u_3\kappa_2^2}{u_2\kappa_1}+\frac{u_3u_4\kappa_2^3}{u_2^2\kappa_1^2}$$

$$-\frac{u_4\lambda_2\kappa_3^2}{u_3\lambda_1\kappa_2}+\frac{u_4\kappa_3^2}{u_3\kappa_2}+2\frac{u_4}{u_2}\frac{\kappa_2\kappa_3}{\kappa_1})\left|\frac{\lambda_1^{-1}\lambda_2^{-1}\mu^{-2}}{u_1^3u_2^2u_3}\right|d\kappa_1 d\kappa_2 d\kappa_3 d\lambda_1 d\lambda_2 d\mu.$$

Patterns for the transfer operator

What do we notice?

1. The transfer $\mathcal{T} : \mathcal{S}(N,\psi\backslash G/N,\psi) \to \mathcal{S}(\frac{G}{G})$ is given by an integral operator in $\frac{n(n-1)}{2} = |\Phi_G^+|$ variables. This is not Fourier transform on $\frac{G}{G}$ viewed as a vector space — something else is going on!
2. If we ignore some factors in the integrand, we get a *multiplicative Fourier convolution along all positive coroots*:

$n = 3$:

$$\mathcal{T}f(u_1,u_2,u_3) = \int_{F^3} f\left(\begin{pmatrix} & & -\lambda\mu u_3 \\ & \kappa_1\lambda^{-1}u_2 & \\ -\kappa_1^{-1}\mu^{-1}u_1 & & \end{pmatrix}\right) \cdot \psi\left(\kappa_1+\lambda+\mu+\cancel{\frac{u_3}{u_2}\frac{\lambda^2}{\kappa_1}+\frac{u_2}{u_1}\frac{\kappa_1^2}{\lambda}}\right)\cdot\left|\frac{\lambda^{-1}}{u_1^2u_2}\right| d\kappa d\lambda d\mu.$$

$n = 4$:

$$\mathcal{T}f(u_1,u_2,u_3,u_4) =$$

2. (cont.) Hence, the operator is a deformation of a *multiplicative Fourier convolution along coroots* for the right action of the diagonal torus on the antidiagonal one:

$$\prod_{\alpha \in \Phi_G^+} \mathcal{F}_{\check{\alpha}},$$

where, for every cocharacter $\check{\lambda}$, $\mathcal{F}_{\check{\lambda}} f(x) := \int f(x\check{\lambda}(\kappa^{-1}))\psi(\kappa)d\kappa$.
Why should these convolutions be relevant? We can deform G to $G_\emptyset = T^{\mathrm{diag}}(N^- \times N)\backslash G^2$, e.g., for $n = 2$, this is the space of 2×2-matrices of rank 1.
Similarly, deform the character of the Whittaker model to the trivial one. Then, the simplified operators *become transfer operators for a comparison*

$$N\backslash G/N \leftrightarrow (G_\emptyset \times G_\emptyset)/G^{2,\mathrm{diag}},$$

with appropriate L-values inserted. The $\alpha \in \Phi_G^+$ appearing here are related to $L(\mathrm{Ad}, 1)$ inserted into the Kuznetsov formula!

3. Even without the simplification, the operators "in principle" satisfy Poisson summation (at least up to $n = 4$), e.g., for $n = 4$, up to multiplication by absolute values, the transformation can be written as

$$\mathcal{F}_{\check{\alpha}_1}\mathcal{F}_{\check{\alpha}_2}\psi(\alpha_1-\alpha_2+\alpha_2\beta_2)\mathcal{F}_{\check{\alpha}_3}\psi(\alpha_3)\mathcal{F}_{\check{\beta}_1}\psi(\alpha_2+2\beta_2)\mathcal{F}_{\check{\beta}_2}\psi(-\alpha_1-\alpha_3)\mathcal{F}_{\check{\gamma}}$$

(up to the appropriate absolute values/Jacobians etc.).

Conclusions from the local results:

1. In a variety of settings, there are transfer operators $\mathcal{T} : \mathcal{S}(\mathfrak{Y}) \to \mathcal{S}(\mathfrak{X})$ corresponding to a map ${}^LG'_Y \to {}^LG_X$ between L-groups. (Mostly ${}^LG'_Y \xrightarrow{\sim} {}^LG_X$, for now.)
2. There are formulas in terms of abelian Fourier convolutions.
3. Although the formulas are not well-understood, they are deformations of well-understood formulas when we let the spaces degenerate.

Is there a better explanation for the formulas? Quantization! (Next summer...)

Global comparisons

Let $G = \mathrm{GL}_n$. We follow the method of Jacquet–Zagier, to write the non-invariant Arthur–Selberg trace formula as a residue of a relative trace formula:

$$\mathrm{TF}(f) = \mathrm{Res}_{s=1} \int K_{f'}(x,x) E_\Phi(x,s) dx,$$

E_Φ an Eisenstein series constructed from a test function Φ on the *Rankin–Selberg* variety $X = k^n \times^{\mathrm{GL}_n^{\mathrm{diag}}} \tilde{G}$, where $\tilde{G} = \mathrm{GL}_n \times \mathrm{GL}_n$.

Here, f is the restriction of $f' \star \hat{\Phi}$ to the zero section of X ($\simeq \mathrm{GL}_n$), and the non-invariant part of the trace formula is defined in a slightly different way than usual.

($\exists$ relevant, but non-overlapping work of Liyang Yang.)

Theorem (S.–Chen Wan; in preparation)
There is a decomposition $\mathrm{TF}(f) = \bigoplus_{j=1}^{n} \mathrm{TF}_j(f)$, *with the following properties:*

- *the summand* TF_j *decomposes spectrally in terms of Arthur parameters of tempered length* j, *that is, the centralizer of the image of Arthur's* SL_2 *has rank* j;
- *the most tempered summand* TF_n *is equal to*

$$\mathrm{Res}_{s=0}\mathrm{KTF}(\mathcal{T}_s^{-1}(f)),$$

where $\mathcal{T}_s$ *is a deformation of the transfer operator* $\mathcal{T} : \mathcal{S}^{-}_{L(\mathrm{Ad},1)}((N,\psi)\backslash G/(N,\psi)(\mathbb{A})) \xrightarrow{\sim} \mathcal{S}(\frac{G}{G}(\mathbb{A}))$ *described before.*

Remarks: 1) The operator $\mathcal{T}_s$ really depends on the data Φ on the Rankin–Selberg variety, but its specialization at $s = 0$ is equal to $\mathcal{T}$.
2) Although we prove it indirectly, "descending" the Rankin–Selberg unfolding to $\mathrm{GL}_n^{\mathrm{diag}}$-orbital integrals, the formula looks like a Poisson summation formula for the transfer operator $\mathcal{T}$, with "boundary terms" TF_j picking up the nontempered parts of the spectrum.

3. For $n = 2$, where $\mathcal{T}$ is simply a one-dimensional Fourier transform this is a variant of the Poisson summation formula of Altuğ.
4. For $n \geq 2$, we see that the isolation of the (conjecturally) tempered spectrum requires a Fourier transform in $\frac{n(n-1)}{2}$ dimensions, according to our calculation of the transfer operator $\mathcal{T}$ (at least, in low ranks).
5. A similar Poisson summation formula was previously proven by S. (directly!) for the "beyond endoscopy" comparison

$$(N, \psi)\backslash G/(N, \psi) \leftrightarrow X \times X/G$$

for $G = \mathrm{PGL}_2$, $X = T\backslash G$, to give a new proof of Waldspurger's formula for toric periods.

In the last few slides, we will discuss problems posed by these global Poisson summation formulas.

Poisson summation formulas

$\rho : {}^L G_X \to {}^L G'_Y, \mathfrak{Y} = Y \times Y/G', \mathfrak{X} = X \times X/G$. It is helpful to stick to the case $\xrightarrow{\sim}$, until we can answer some questions there.
Suppose (*huge assumption!*) that we knew the local transfer operator $\mathcal{T}_\rho : \mathcal{S}(\mathfrak{Y}) \to \mathcal{S}(\mathfrak{X})$ in terms of Fourier transforms and other "elementary" operations (such as multiplication by the absolute value of a rational function) which, *in principle*, satisfy a Poisson summation formula. *Can we prove such a formula?*

A technique that has been used is to introduce a complex parameter s and *deform* the spaces and operators in a meromorphic way, trying to prove a PSF for a deformed operator

$$\mathcal{T}_{\rho,s} : \mathcal{S}(\mathfrak{Y})_s \to \mathcal{S}(\mathfrak{X})_s$$

when $\Re(s) \gg 0$. (Since we are introducing L-functions, this s should be related to the point of their evaluation.)
Still, it is unclear in rank > 1 how to prove such a PSF directly.
Problem: Understand deformations of spaces of orbital integrals and transfer operators, and use them to prove PSF.

The role of the functional equation/Hankel transforms

Similarly, we will have a spectral decomposition "for large values of s", but since we are introducing L-values outside of the domain of convergence, it will not directly apply to the desired s. Thus, even if we have a Poisson summation formula for $\Re(s) \gg 0$, and meromorphic continuation, we don't have a spectral decomposition! Rather than use hard analytic number theory to meromorphically continue the spectral decomposition, a technique that I have used successfully is to use the functional equation + Phragmén–Lindelöf. But then, it is essential, besides the comparison $\mathcal{T}_\rho : \mathcal{S}(\mathfrak{Y}) \to \mathcal{S}(\mathfrak{X})$, to have a comparison $\mathcal{S}(\mathfrak{Y})_s \leftrightarrow \mathcal{S}(\mathfrak{Y})_{-s}$ corresponding to the functional equation of the pertinent L-function.

- In the setting of the KTF, studied by Herman for $r = \mathrm{Std}$ of GL_2 ($\exists$ local formulas of Jacquet for Std, S. for Sym^2).
- In the setting of the STF, the operator giving rise to the functional equation has been called *Hankel transform* by B.C. Ngô, who has been studying it, expanding on ideas from the papers of Braverman and Kazhdan on γ-factors.

The role of the functional equation/Hankel transforms

Thus, closely related to the problem of understanding transfer operators and proving Poisson summation formulae for them, we have

Problem: For $\mathfrak{Y} =$ the Kuznetsov or STF quotient, and $r : {}^LG \to \mathrm{GL}(V)$, understand the local Hankel transforms

$$\mathcal{S}^-_{L(r,\frac{1}{2}+s)}(\mathfrak{Y}) \xrightarrow{\sim} \mathcal{S}^-_{L(r,\frac{1}{2}-s)}(\mathfrak{Y}),$$

and prove a Poisson summation formula for them.

Interestingly, we are coming full cirle: Langlands functoriality was meant to prove analytic properties of L-functions, but, if the answer lies in Beyond Endoscopy, it seems that we will directly need to prove those properties along the way!

Thank you!

PS: Hope to see you in person soon!