Adams' conjecture on theta correspondence

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Introduction

- In his paper
  L-functoriality for Dual Pairs, Asterisque 171-172, 1989,85-129,
  J. Adams loosely formulates a conjecture regarding representations in local Arthur packets. He predicts that their theta lifts on the groups of bigger rank are going to be, in certain situations, also members of Arthur packets of similar form.

- This prediction was made more precise for irreducible admissible representations of classical p-adic groups (more precisely, for p-adic symplectic-even orthogonal dual reductive pairs) by Mœglin in her paper *Conjecture d'Adams pour la correspondance de Howe et filtration de Kudla,* Arithmetic geometry and automorphic forms, 445–503, Adv. Lect. Math. (ALM), 19, Int. Press, Somerville, MA, 2011
- There, she resolved the case of discrete series representations
- She also posed two important questions which we address later

Overview of the talk

- brief overview of local Arthur packets for symplectic and even orthogonal p-adic groups
- brief overview on local theta correspondence for p-adic classical groups
- the statement of the conjecture; questions posed by Mœglin
- our results in the case of discrete diagonal restriction case

Notation-groups and representations

- $G=$ a classical group defined over a p-adic field F of characteristic zero: symplectic, even orthogonal, unitary-those which have Arthur endoscopic classification and Mœglin explicit construction of the local packets (in a uniform way); the case of metaplectic groups can also be included when the transfer is known. (Mœglin discusses the extent of applicability of her local construction in Multiplicite 1 dans paquets d'Arthur aux places p-adiques, Shahidi's volume)
- **parabolic subgroups**
- V_m a quadratic space/skew-symmetric/hermitian space of dimension m; $G = G(V_m)$ the (F-points of) corresponding isometry group
- a subset $\{v_1, \ldots, v_r, v_1', \ldots, v_r'\}$ of V_m such that $(v_i, v_j) = (v_i', v_j') = 0$ and $(v_i, v_j') = \delta_{ij}$ ($r-$ the Witt index of V_m
- $B = TU$ the standard F-rational Borel subgroup of $G(V_m)$, i.e. the subgroup of $G(V_m)$ stabilizing the flag

$$0 \subset \mathrm{span}\{v_1\} \subset \mathrm{span}\{v_1, v_2\} \subset \cdots \subset \mathrm{span}\{v_1, v_2, \ldots, v_r\},$$

- $t \leq r$ we set $U_t = \text{span}\{v_1, \ldots, v_t\}$ and $U'_t = \text{span}\{v'_1, \ldots, v'_t\}$; $\rightsquigarrow$ the decomposition

$$V_m = U_t \oplus V_{m-2t} \oplus U'_t$$

- The subgroup Q_t of G which stabilizes U_t is a maximal parabolic subgroup of G; a Levi decomposition $Q_t = M_t N_t$, where $M_t = \text{GL}(U_t) \times G(V_{m-2t})$ is the Levi component (stabilizes U'_t)
- t varies $\rightsquigarrow$ the set $\{Q_t : t \in \{1, \ldots, r\}\}$ of standard maximal parabolic subgroups.
- Further partitioning $t \rightsquigarrow$ the rest of the standard parabolic subgroups
- the Levi factor of a standard parabolic subgroup is of the form

$$\text{GL}_{t_1}(F) \times \cdots \times \text{GL}_{t_k}(F) \times G(V_{m-2t}) \quad (t = t_1 + \cdots + t_k)$$

(for the symplectic groups analogously; unitary groups -with E (a quadratic extension of F) in place of F)

- $\mathrm{Ind}_P^G(\tau_1 \otimes \cdots \otimes \tau_k \otimes \pi_0)$, where τ_i is a smooth representation of $\mathrm{GL}_{t_i}(F)$, $i = 1, \ldots, k$, and π_0 is a smooth representation of $G(V_{n-2t})$ (with $t = t_1 + \cdots + t_k$) is denoted by (Zelevinsky notation)
$$\tau_1 \times \cdots \times \tau_k \rtimes \pi_0.$$

- similarly for general linear groups

- For a *segment* of cuspidal representations (of $GL(n)$)$\{\nu^a\rho, \ldots, \nu^b\rho\}$ (so that $a \geq b$ or vice versa) we denote by $\langle \nu^a\rho, \ldots, \nu^b\rho \rangle$ the unique irreducible subrepresentation of the representation
$$\nu^a\rho \times \cdots \times \nu^b\rho,$$
thus, if $a \geq b$ this is the Steinberg representation (attached to that segment); otherwise, this is generalized trivial representation.

Local Arthur packets

- F p-adic field of characteristic zero
- $GL(n) = GL(n, F)$; the representations in "Arthur class for $GL(n)$"=local components of the discrete spectrum of automorphic representation of $GL(n)$ (Mœglin-Waldspurger)
- explicit description of Arthur class for $GL(n)$: ρ_i-an irreducible, smooth, cuspidal unitary representation of $GL(n_{\rho_i}), i = 1, 2, \ldots, k$
- $a_i, b_i \in \mathbb{Z}_{>0},\ l_i \in \mathbb{Z}_{>0}$.
- $St(\rho_i, a_i)$=Steinberg representation=the unique irreducible subrepresentation of

$$\rho_i \nu^{\frac{a_i-1}{2}} \times \rho_i \nu^{\frac{a_i-1}{2}-1} \times \cdots \times \rho_i \nu^{-\frac{a_i-1}{2}}$$

- $Sp(St(\rho_i, a_i), b_i)$=Speh representation=is the unique irreducible subrepresentation of

$$St(\rho_i, a_i)\nu^{-\frac{b_i-1}{2}} \times St(\rho_i, a_i)\nu^{-\frac{b_i-1}{2}+1} \times \cdots \times St(\rho_i, a_i)\nu^{\frac{b_i-1}{2}}.$$

- The Arthur class for $GL(n)$ consist of all the irreducible representations of the form

$$\times_{i=1}^{k} Sp(St(\rho_i, a_i), b_i)^{l_i},$$

  where $Sp(St(\rho_i, a_i), b_i)^{l_i}$ denotes the product $Sp(St(\rho_i, a_i), b_i) \times Sp(St(\rho_i, a_i), b_i) \times \cdots \times Sp(St(\rho_i, a_i), b_i)$ (l_i times) for all such triples a_i, b_i, l_i such that $\sum_{i=1}^{k} a_i b_i n_{\rho_i} l_i = n$ holds.
- LLC for $GL(n)$

$$\{\text{(eq.classes) irreducible unitary supercuspidal reps. of } GL(n)\}$$

$$\leftrightarrow$$

$$\{n-\text{dimensional irreduc. unitary reps. of the Weil group } W_F\}$$

  this leads to

Parameters for the Arthur class for $GL(n)$

- equivalence classes of parameters

$$\psi : W_F \times SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to GL(n,\mathbb{C})$$

- such that $\psi_{|W_F}$ is unitary (and some other requirements...)
- $\psi_{|SL(2,\mathbb{C})\times SL(2,\mathbb{C})}$ is algebraic
- For each $a \in Z_{>0}$, the unique irreducible algebraic representation of $SL(2,\mathbb{C})$ of dimension a is denoted by ν_a. (corresponds to a_i or b_i in the Speh representation above)
- Historically, the first $SL(2,\mathbb{C})$ comes from the monodromy operator, and the second $SL(2,\mathbb{C})$ from non-temperedness of the representations in Arthur class
- $GL(n)$ representation (as above) corresponding to this parameter ψ denoted by π_ψ

Arthur parameters for symplectic/full orthogonal groups

- we specify the general construction to the case at hand;
- ($\hat{G}$-equivalence class of) continuous, unitary, algebraic homomorphisms

$$\psi : W_F \times SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to O(l,\mathbb{C})$$

- if l is odd with values in $SO(l,\mathbb{C})$
- if l is even, the restriction of ψ to W_F composed with the determinant of $O(l,\mathbb{C})$ gives a character of W_F which corresponds to η_V via the class field theory (where $G = O(V_{\eta_V})$).
- Here, the L-group for the symplectic group can be identified with the corresponding dual group; and for the (special) even orthogonal group with the corresponding discriminant which is not a square in F^*, we can identify $SO(2n,\mathbb{C}) \rtimes Gal(E/F) \cong O(2n,\mathbb{C})$).
- The set of all equivalence classes of A-parameters for G is denoted by $\Psi(G)$.

- We can decompose ψ as above
$$\psi = \oplus_{i=1}^{k} l_i(\rho_i \otimes \nu_{a_i} \otimes \nu_{b_i})$$
- $\rightsquigarrow$
$$\mathrm{Jord}(\psi) = \{(\rho_i, \nu_{a_i}, \nu_{b_i}) \text{ with multiplicity } l_i,\ i = 1, 2, \ldots, k\}$$
(a multiset)
- selfdual $\rho = n_\rho$-dimensional irreducible unitary continuous rep of W_F ($\leftrightarrow$ selfdual irreducible smooth cuspidal representation of $GL(n_\rho, F)$) is *orthogonal* if it factors through $O(n_\rho, \mathbb{C})$, i.e. if $L(s, \rho, Sym^2)$ has a pole at $s = 0$.
- other possibility for such ρ is that it is of *symplectic* type (i.e. $L(s, \rho, \Lambda^2)$ has a pole at $s = 0$.)

- $(\rho_i, \nu_{a_i}, \nu_{b_i})$ is of orthogonal type if $\rho_i \otimes \nu_{a_i} \otimes \nu_{b_i}$ factors through an orthogonal group $\leftrightarrow$ $a_i + b_i$ is even if ρ_i is of orthogonal type or $a_i + b_i$ is odd if ρ_i is of symplectic type
- analogously we say that $(\rho_i, \nu_{a_i}, \nu_{b_i})$ is of symplectic type
- let ψ_p the sum of all the summands in ψ which are of the same parity as $\hat{G}$; in our case, this means of orthogonal parity; the other summands are gathered ("half" of them!-with ρ_i non-selfdual or ρ_i selfdual but $(\rho_i, \nu_{a_i}, \nu_{b_i})$ of symplectic type) in ψ_{np} so that
$$\psi = \psi_{np} \oplus \psi_p \oplus \psi_{np}^\vee$$
- ψ_p is part "of good parity"

- to define Arthur packet attached to this parameter, we need to introduce various centralizers of ψ in the corresponding L-group and its quotients; without precision, we denote the relevant centralizer by $\mathcal{S}_\psi$. Now the characters of $\mathcal{S}_\psi$ parametrize representations $\pi(\psi,\epsilon)$ ($\epsilon \in \widehat{\mathcal{S}_\psi}$) of G,
- $\pi(\psi,\epsilon)$ is semisimple, admissible, of finite length, (maybe zero) such that the following holds

Theorem (Arthur)

The character

$$\sum_{\epsilon\in\widehat{\mathcal{S}_\psi}} \epsilon\left(\psi(1,1,\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix})\right) tr\pi(\psi,\epsilon)$$

is a stable distribution; moreover, it is the transfer of the (twisted invariant) character of the representation π_ψ.

- Precise (inductive) definition of representations $\pi(\psi, \epsilon)$ is given by Mœglin, and Mœglin-Waldspurger
- Internal parametrization of representations inside A-packet differs between Arthur and Mœglin; the exact correspondence between parameterizations is given by Xu.
- Let Π_ψ be the (multi)set of all irreducible subrepresentations of all $\pi(\psi, \epsilon)$, $\epsilon \in \widehat{\mathcal{S}_\psi}$ ($\rightsquigarrow$ "Arthur packet")
- We have the following

Theorem (Moeglin)

The set Π_ψ is multiplicity free.

A-parameters for G with discrete diagonal restriction

- Assume $\psi = \oplus_{i=1}^{k} l_i(\rho_i \otimes \nu_{a_i} \otimes \nu_{b_i}) \in \Psi(G)$
- $\psi \rightsquigarrow \psi_d := \psi \circ \Delta$, where $\Delta : SL(2,\mathbb{C}) \to SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ is the diagonal embedding
- Set $B_i := \frac{|a_i - b_i|}{2},\ A_i = \frac{a_i + b_i}{2} - 1,\ \zeta_i = sign(a_i - b_i)$ if $a_i \neq b_i$ and equal to 1 if $a_i = b_i$. Then
- $$\psi_d = \oplus_{i=1}^{k} l_i \left(\oplus_{j_i \in [B_i, A_i]} \rho_i \otimes \nu_{2j_i+1} \right)$$
- ψ is of discrete diagonal restriction if $\psi = \psi_p$ and ψ_d is multiplicity free (i.e, $\forall i,\ l_i = 1$ and, if, for $i \neq j$ we have $\rho_i \cong \rho_j$, then $[B_i, A_i]$ and $[B_j, A_j]$ are disjoint).

Elementary A-packets

- Assume $\psi = \psi_p = \oplus_{i=1}^{k} \rho_i \otimes \nu_{a_i} \otimes \nu_{b_i}$ is of discrete diagonal restriction (DDR); then ψ is elementary if, for all $i = 1, 2, \ldots, k \ \min\{a_i, b_i\} = 1$

Theorem (Mœglin)

Assume ψ is elementary A-parameter. Then, for every $\epsilon \in \hat{\mathcal{S}}_\psi$, $\pi(\psi, \epsilon)$ is non-zero and irreducible.

- Here, $\mathcal{S}_\psi$ is very easy to describe...
- Mœglin has two different constructions of $\pi(\psi, \epsilon)$ for ψ elementary: one "mimicking" the gradual construction (by induction over rank) of the discrete series representations (with some adjustments and sensitive parts); thus boiling down to the case of cuspidal representations. The other one starts from the discrete series A-packets and uses partial Aubert involution.

Theorem (Mœglin)

Suppose that $\psi \in \Psi(G)$ is DDR. Then, for each $\epsilon \in \hat{\mathcal{S}}_\psi$, $\pi(\psi, \epsilon) \neq 0$ and

$$\pi(\psi, \epsilon) = \oplus_{(t,\eta):\epsilon=\epsilon_{t,\eta}} \pi(\psi, t, \eta),$$

where $\pi(\psi, t, \eta)$ is an irreducible representation which can be inductively described and (t, η) are functions on $\mathrm{Jord}(\psi)$ described in the following way:

$$t : \mathrm{Jord}(\psi) \to \mathbb{Z}_{\geq 0} \text{ such that } t(\rho, a_i, b_i) \leq \frac{\min\{a_i, b_i\}}{2}, \forall (\rho, a_i, b_i)$$

and

$$\eta : \mathrm{Jord}(\psi) \to \{1, -1\}.$$

We require that $\eta(\rho, a_i, b_i) = 1$ if $t(\rho, a_i, b_i) = \frac{\min(a_i,b_i)}{2}$. To such a pair, one attaches a function $\epsilon_{t,\eta} : \mathrm{Jord}(\psi) \to \{1, -1\}$ in the following way

$$\epsilon_{t,\eta}(\rho, a_i, b_i) = \eta(\rho, a_i, b_i)^{\min(a_i,b_i)}(-1)^{\lfloor \frac{\min(a_i,b_i)}{2} \rfloor + t(\rho,a_i,b_i)}$$

- the representations $\{\pi(\psi, t, \eta) : \epsilon_{t,\eta} \in \hat{\mathcal{S}}_\psi\}$, thus, form the DDR A-packets, and we shall test Adams conjecture on them
- They include elementary A-packets, and their construction is inductive, starting point being elementary packets (corresponding to the situation of $t \equiv 0$)
- General A-packets of good parity are given through taking Jacquet modules of certain DDR A-packets which "dominate" them; the chosen order on the summands is very important here
- To prove Adams conjecture, although for DDR packets, we encounter non-DDR A-packets

The construction of representations $\pi(\psi, t, \eta)$

- let $\psi = \oplus_{i=1}^{k} \rho_i \otimes \nu_{a_i} \otimes \nu_{b_i} \in \Psi(G)$ be DDR parameter
- The construction is through a recursive procedure on $l(\psi) = \sum_{i=0}^{k} (\min(a_i, b_i) - 1)$
- Assume first that $l(\psi) = 0$. Then we have elementary parameter (then, necessarily, by the definition $t(\rho, a_i, b_i) = 0$, $\forall i = 1, 2, \ldots, k$ and $\pi(\psi, t, \eta) = \pi(\psi, \epsilon_{t,\eta})$
- Assume that there exists $i_0 \in \{1, 2, \ldots, k\}$ such that $\min(a_{i_0}, b_{i_0}) > 1$ (note that it is irrelevant which i_0 we choose if there are more of them)
- If $t(\rho_{i_0} \otimes \nu_{a_{i_0}} \otimes \nu_{b_{i_0}}) = 0$ then we construct another parameter ψ' in the following way

The construction of representations $\pi(\psi, t, \eta)$-continued

instead of $\rho_{i_0} \otimes \nu_{a_{i_0}} \otimes \nu_{b_{i_0}}$, in ψ' we put the sum

$$\rho_i \otimes \nu_{a_{i_0}-b_{i_0}+1} \otimes \nu_1 \oplus \rho_i \otimes \nu_{a_{i_0}-b_{i_0}+3} \otimes \nu_1 \oplus \cdots \oplus \rho_i \otimes \nu_{a_{i_0}+b_{i_0}-1} \otimes \nu_1, \text{ if } \zeta_{i_0} = 1,$$

or

$$\rho_i \otimes \nu_1 \otimes \nu_{a_{i_0}-b_{i_0}+1} \oplus \rho_i \otimes \nu_1 \otimes \nu_{a_{i_0}-b_{i_0}+3} \oplus \cdots \oplus \rho_i \otimes \nu_1 \otimes \nu_{a_{i_0}+b_{i_0}-1}, \text{ if } \zeta_i = -1.$$

The rest of the summands remain the same, with t', η' defined on $\mathrm{Jord}(\psi')$ in the following way:

$$t'_{|\mathrm{Jord}(\psi)\backslash(\rho_{i_0}, \nu_{a_{i_0}}, \nu_{b_{i_0}})} = t_{|\mathrm{Jord}(\psi)\backslash(\rho_{i_0}, \nu_{a_{i_0}}, \nu_{b_{i_0}})},$$

$$\eta'_{|Jord(\psi)\backslash(\rho_i, \nu_{a_{i_0}} \otimes \nu_{b_{i_0}})} = \eta_{|Jord(\psi)\backslash(\rho_{i_0}, \nu_{a_{i_0}}, \nu_{b_{i_0}})}.$$

$$t'(\rho_{i_0}, c, 1) = 0, \forall c \in [a_{i_0} - b_{i_0} + 1, a_{i_0} + b_{i_0} - 1], \text{ if } \zeta_{i_0} = 1,$$

and

$$t'(\rho_{i_0}, 1, c) = 0, \forall c \in [a_{i_0} - b_{i_0} + 1, a_{i_0} + b_{i_0} - 1], \text{ if } \zeta_{i_0} = -1.$$

The construction of representations $\pi(\psi, t, \eta)$-continued II

and

$$\eta'(\rho_{i_0}, c, 1) = \eta(\rho_{i_0}, a_{i_0}, b_{i_0})(-1)^{\frac{c-|a_{i_0}-b_{i_0}|-1}{2}}, \text{ if } \zeta_{i_0} = 1,$$

$$\eta'(\rho_{i_0}, 1, c) = \eta(\rho_{i_0}, a_{i_0}, b_{i_0})(-1)^{\frac{c-|a_{i_0}-b_{i_0}|-1}{2}}, \text{ if } \zeta_{i_0} = -1.$$

Then, $\pi(\psi, t, \eta) = \pi(\psi', t', \eta')$ and

- if $t(\rho_{i_0} \otimes \nu_{a_{i_0}} \otimes \nu_{b_{i_0}}) > 0$, then we construct another parameter ψ' in the following way: we replace $(\rho_{i_0}, \nu_{a_{i_0}}, \nu_{b_{i_0}})$ with $(\rho_{i_0}, \nu_{a_{i_0}}, \nu_{b_{i_0}-2})$ if $\zeta = 1$ and $(\rho_i, \nu_{a_{i_0}-2}, \nu_{b_{i_0}})$ if $\zeta = -1$. We define $t'(\rho_{i_0}, \nu_{a_{i_0}}, \nu_{b_{i_0}-2})$ (or $t'(\rho_{i_0}, \nu_{a_{i_0}-2}, \nu_{b_{i_0}})$) to be equal to $t(\rho_{i_0}, \nu_{a_{i_0}}, \nu_{b_{i_0}}) - 1$. Then, $\pi(\psi, t, \eta)$ is the unique irreducible subrepresentation of

$$\langle \nu^{\frac{a_{i_0}-b_{i_0}}{2}} \rho_{i_0}, \ldots, \nu^{-\zeta_{i_0}(\frac{a_{i_0}+b_{i_0}}{2}-1)} \rho_{i_0} \rangle \rtimes \pi(\psi', t', \eta').$$

- The mapping $(t, \eta) \mapsto \pi(\psi, t, \eta)$ is injective

The order on $\mathrm{Jord}(\psi)$

- Let $\psi \in \Psi(G)$ be a parameter of good parity
- To be able to define non-DDR packets one needs to introduce certain order on $\mathrm{Jord}(\psi)$
- This is order has to satisfy certain conditions (we do not express them); but these condition do not specify the order uniquely
- We use the following order: note $(\rho, a, b) \rightsquigarrow (\rho, A, B, \zeta)$ we defined above. Assume that $(\rho, A, B, \zeta) \neq (\rho, A', B', \zeta')$. Then, if $(\rho, A, B, \zeta) > (\rho, A', B', \zeta')$ we have that $B > B'$ or $B = B'$ and $A > A'$ or $B = B'$ and $A = A'$ and $\zeta = 1$.
- If ψ is DDR, then there is an obvious order on $\mathrm{Jord}(\psi)$ (since $[B', A']$ and $[B, A]$ are disjoint) which (obviously) satisfies these conditions
- From now on, for a DDR parameter, we write its summands in this (increasing) order (for some fixed ρ_i; the order between different ρ's is not important), and, if the parameter is not DDR, we choose the order above

Brief overview of theta correspondence

- Fix a non-trivial additive character of F, $\psi_F : F \to \mathbb{C}, \rightsquigarrow$ the corresponding Weil representation ω_{W,ψ_F} of the metaplectic group $Mp(W)$, where W is a p-adic symplectic space
- symplectic/even-orthogonal reductive pair (or vice versa; symmetric approach) $G(W_n) \times H(V_m)$ in $Sp(W_n \otimes V_m)$ (m, n dimensions of the corresponding spaces; denote $\epsilon_0 = 1$ if W_n is symplectic, $\epsilon_0 = -1$ if W_n is orthogonal)
- need to fix a splitting $G(W_n) \times H(V_m) \to \mathrm{Mp}(W_n \otimes V_m); \rightsquigarrow$ fix a pair of characters (χ_V, χ_W) of F^* attached to the spaces W_n, V_m. (these characters can be chosen to be constant within a fixed Witt tower, which justifies the omitting m and n and using χ_W, χ_V to denote them). These characters, along with the non-trivial additive character $\psi_F : F \to \mathbb{C}$, determine a splitting $G(W_n) \times H(V_m) \to \mathrm{Mp}(W_n \otimes V_m)$
- can restrict ω_{W,ψ_F} to $G(W_n) \times H(V_m) \rightsquigarrow \omega_{W_n,V_m,\psi_F}$

- For any $\pi \in \mathrm{Irr}(G(W_n))$, the maximal π-isotypic quotient of $\omega_{m,n}$ is of the form
$$\pi \otimes \Theta(\pi, V_m),$$
- $\Theta(\pi, V_m)$ "big theta lift"; has a unique irreducible quotient $\theta(\pi, V_m)$ (irreducible smooth representation of V_m) "small theta lift" (conjectured by Howe; proved by Waldspurger, Gan, Takeda) $\rightsquigarrow$ *theta correspondence*
- If $\pi \in \mathrm{Irr}(G(W_n))$, denote $l = n + \epsilon_0 - m$; (*the relative index*) then $\theta_l(\pi) := \theta(\pi, V_m)$; useful notation when examining occurrences of the theta lift on the same tower

- we examine the lifts of π on the pair of towers simultaneously (e.g., when π is a representation of a symplectic group, we examine its lifts on the a pair of quadratic towers with the same discriminant and different Hasse invariants)

- *the first occurrence index in a tower-the persistence principle; the conservation conjecture* (Sun-Zhu) with respect to the pair of towers as above

- if l' is the first (relative) occurrence index in one tower, and l'' in the other (for given $\pi \in \mathrm{Irr}(G(W_n))$),then $l' + l'' = -2$ (the conservation conjecture)
- We denote by $l(\pi)$ the first occurrence index on the "going down" tower (thus, $l(\pi) \geq -1$)). If π is a representation of an orthogonal group, instead of towers (there is only one symplectic tower!) one examines the lifts of π and $\pi \otimes \det$ simultaneously; note that in the case of symplectic-even orthogonal pair, $l(\pi)$ is odd

Normalization; choices

- for theta correspondence, we have choices ψ_F for the Weil representations χ_V, χ_W for the splittings groups in the dual reductive pair of interest
- in formation of local A-packets-Whittaker normalization of transfer maps (following Arthur, i.e. Xu who makes a comparison)
- in her paper devoted to Adams conjecture Mœglin points out that the parametrization inside one DDR A-packet is fully determined by the parametrization of cuspidal representations (i.e. in $\pi(\psi, t, \eta)$ η could be different)
- the parametrization of cuspidals Mœglin uses is different from ours (we use Arthur parametrization through Atobe and Gan results on theta lifts of tempered representation); this does not effect the stability statements

Adams' conjecture

- Originally, the Adams conjecture is posed for the reductive dual pairs for the real groups
- He observed that the "tail" which comes "high" in the target tower of the theta correspondence can be interpreted as a representation of the second $SL(2, \mathbb{C})$-factor in the Arthur parameter of the original group
- Originally, it was (wrongly) conjectured that this kind of behaviour can be observed for the Langlands parameter, and would be form of functoriality, but it is soon observed that theta correspondence does not respect L-packets (even for discrete series representations)
- He expected that the newly constructed objects (by Barbasch and Vogan) which could be interpreted as Arthur packets for real groups could be the right setting for the conjecture

The statement for the DDR case

Assume that $\psi \in \Psi(G)$ is DDR. Let $\pi(\psi, t, \eta) \in \Pi_\psi$ be a representation of $G(W_n)$. Examine the lifts of $\pi(\psi, t, \eta)$ on a fixed tower (so, if $G(W_n)$ the lifts are in an orthogonal tower, and if $G(W_n)$ is orthogonal, the lifts are in symplectic tower). Then, there exists $r_0 \in \mathbb{Z}_{\geq 0}$ such that, for all $r \geq r_0$ the representation $\theta_{-(2r+1)}(\pi(\psi, t, \eta))$ belongs to A-packet Π_{ψ_r}, where

$$\psi_r = \chi_V^{-1}\chi_W\psi \oplus \chi_V^{-1}\chi_W \otimes \nu_1 \otimes \nu_{2r+1}.$$

Remarks

1. The statement is the same for non-DDR packets (for non-DDR ψ, $\pi(\psi, t, \eta)$ can be zero; it is non-trivial to decide if it is non-zero)
2. It is pretty obvious that such r_0 exists

Mœglin's questions (in Conjecture d'Adams...)

1. For given $\pi(\psi, t, \eta)$, find the smallest r_0 with the property prescribed above
2. Is there any $0 \leq r' < r_0$ such that $\theta_{-(2r+1)}(\pi(\psi, t, \eta)) \in \Pi_{\psi_r}$?

Remarks

- In this paper, Mœglin uses ad-hoc parameterization of cuspidal representations inside an A-packet in which lifts will have "more similar" (t, η) parameters to one of original representations. We use Arthur parametrization.
- Also she does not use necessarily the order on summands we introduced above; the same reasons
- Although we deviate from her choices, in the end, those choices "cancel", and we get elegant results

The results

might want to consider them slightly conjectural just now...

- Let $\pi(\psi, t, \eta) \in \Pi_\psi$, such that ψ is DDR parameter of $G(W_n)$
- we can write it as

$$\psi = \psi' \oplus \psi_{\chi_V},$$

  where ψ_{χ_V} contains all the summands of the form $\chi_V \otimes \nu_* \otimes \nu_*$
- If the theta lift $\theta_{-(2r+1)}(\pi)$ is in the A-packet ψ_r, only the parameters (t, η) on ψ_{χ_V} are going to change; the parameters (t, η) on ψ'–part stay the same.
- Assume (**we respect the order!**)

$$\psi_{\chi_V} = \chi_V \otimes \overset{t_1,\eta_1}{\nu_{a_1}} \otimes \nu_{b_1} \oplus \chi_V \otimes \overset{t_2,\eta_2}{\nu_{a_2}} \otimes \nu_{b_2} \oplus \cdots \oplus \chi_V \otimes \overset{t_k,\eta_k}{\nu_{a_k}} \otimes \nu_{b_k},$$

  where $t_i = t(\chi_V \otimes \nu_{a_i} \otimes \nu_{b_i}),\ \eta_i = \eta(\chi_V \otimes \nu_{a_i} \otimes \nu_{b_i})$.

- Denote by k_0 the largest index such that the sum

$$\chi_V \otimes \nu_{a_1}^{t_1,\eta_1} \otimes \nu_{b_1} \oplus \chi_V \otimes \nu_{a_2}^{t_2,\eta_2} \otimes \nu_{b_2} \oplus \cdots \oplus \chi_V \otimes \nu_{a_{k_0}}^{t_{k_0},\eta_{k_0}} \otimes \nu_{b_{k_0}}$$

  is "*cuspidal*"; i.e. $t_1 = t_2 = \cdots = t_{k_0} = 0$, the corresponding segments $[B_1, A_1], \ldots, [B_{k_0}, A_{k_0}]$ are juxtaposed (so no gaps between each two of them), starting from $B_1 = 0$, and the characters on them (recall the formation of $\chi_V \otimes \nu_{a_i}^{t_i=0,\eta_i} \otimes \nu_{b_i}$) alternate.
- Denote with l_0 the largest index, which is larger than k_0, if k_0 as above exists, otherwise just take the largest l_0 such that for $\chi_V \otimes \nu_{a_{l_0}}^{t_{l_0},\eta_{l_0}} \otimes \nu_{b_{l_0}}$ we have $\zeta_{l_0} = -1$, $t_{l_0} = 0$. Then we have

Theorem

$l(\pi(\psi, t, \eta)) = a_{l_0} + b_{l_0} - 1$, if l_0 described above exists. If it does not exist, then $l(\pi(\psi, t, \eta)) = a_{k_0} + b_{k_0} - 1$. If both l_0 and k_0 do not exist, then $l(\pi(\psi, t, \eta)) = -1$.

The results-continuation

We keep the notation as above ($\pi(\psi, t, \eta)$ is a representation of $G(W_n)$; we lift on both target towers simultaneously)

Theorem

- *For $2r+1 > a_k + b_k - 1$, $\theta_{-(2r+1)}(\pi(\psi, t, \eta)) \neq 0$ (previous theorem), belongs to Π_{ψ_r} and we have $\theta_{-(2r+1)}(\pi(\psi, t, \eta)) \ldots$*

$$
\begin{aligned}
&\chi_V^{-1}\chi_W\psi' \oplus \\
&\chi_V^{-1}\chi_W \overset{t_1,-\eta_1}{\otimes} \nu_{a_1} \otimes \nu_{b_1} \oplus \chi_V^{-1}\chi_W \overset{t_2,-\eta_2}{\otimes} \nu_{a_2} \otimes \nu_{b_2} \oplus \cdots \oplus \\
&\chi_V^{-1}\chi_W \overset{t_k,-\eta_k}{\otimes} \nu_{a_k} \otimes \nu_{b_k} \oplus \chi_V^{-1}\chi_W \overset{\epsilon}{\otimes} \nu_1 \otimes \nu_{2r+1}
\end{aligned}
$$

where ϵ depends on the Hasse invariant of the tower.

Theorem (continuation)

- *Now we go down with respect to* r*. As long as* $2r+1 > l_0, k_0$ *(i.e.* $\zeta_k = 1,\ \zeta_{k-1} = 1$ *,etc., or* $\zeta_k = -1$ *with* $t_k > 0$ *etc.) we can jump "between, inside and around" each such summand in the following way: assume first that* $k > l_0$*. Then,* $\theta_{-(2r+1)}(\pi(\psi, t, \eta)) \in \Pi_{\psi_r}$ *for all* r *such that* $|a_k - b_k| + 1 \leq 2r+1 \leq a_k + b_k - 1$ *where we have the following* $\theta_{-(2r+1)}(\pi(\psi, t, \eta)) \ldots$

$$\chi_V^{-1}\chi_W\psi' \oplus$$

$$\chi_V^{-1}\chi_W \overset{t_1,-\eta_1}{\otimes} \nu_{a_1} \otimes \nu_{b_1} \oplus \chi_V^{-1}\chi_W \overset{t_2,-\eta_2}{\otimes} \nu_{a_2} \otimes \nu_{b_2} \oplus \cdots \oplus$$

$$\chi_V^{-1}\chi_W \overset{t_k,-\eta_k}{\otimes} \nu_{a_k} \otimes \nu_{b_k} \oplus \chi_V^{-1}\chi_W \overset{\epsilon}{\otimes} \nu_1 \otimes \nu_{2r+1}$$

if $|a_k - b_k| + 1 < 2r+1 \leq a_k + b_k - 1$ *(note that* ψ_r *is not DDR any more)*

Theorem (continuation II)

For $2r+1 = |a_k - b_k| + 1$ *the rest of the parameter of* $\theta_{-(2r+1)}(\pi(\psi, t, \eta))$ *remains the same, but instead of*

$$\chi_V^{-1}\chi_W \overset{t_k,-\eta_k}{\otimes} \nu_{a_k} \otimes \nu_{b_k} \oplus \chi_V^{-1}\chi_W \overset{\epsilon}{\otimes} \nu_1 \otimes \nu_{|a_k-b_k|+3}$$

we have

- $$\chi_V^{-1}\chi_W \overset{-\epsilon(-1)^{b_k-1}}{\otimes} \nu_1 \otimes \nu_{a_k-b_k+1} \oplus \chi_V^{-1}\chi_W \overset{t_k,\eta_k}{\otimes} \nu_{a_k} \otimes \nu_{b_k}, \text{ if } \zeta_k = 1,$$

  $$0 \leq t_k \leq \lfloor \frac{b_k-1}{2} \rfloor$$

- $$\chi_V^{-1}\chi_W \overset{\epsilon}{\otimes} \nu_1 \otimes \nu_{a_k-b_k+1} \oplus \chi_V^{-1}\chi_W \overset{t_k,-\eta_k}{\otimes} \nu_{a_k} \otimes \nu_{b_k}, \text{ if } \zeta_k = 1, \ t_k = \frac{b_k}{2}$$

Theorem

- $$\overbrace{\chi_V^{-1}\chi_W \otimes \nu_1 \otimes \nu_{a_k-b_k+1}}^{-\epsilon(-1)^{a_k-1}=-\eta_k} \oplus \overbrace{\chi_V^{-1}\chi_W \otimes \nu_{a_k} \otimes \nu_{b_k}}^{t_k+1,\eta_k},\ \textit{if}\ \zeta_k=-1,$$

  $$\epsilon=-\eta_k(-1)^{a_k-1},\ 0<t_k<\lfloor\frac{a_k-1}{2}\rfloor$$

- $$\overbrace{\chi_V^{-1}\chi_W \otimes \nu_1 \otimes \nu_{a_k-b_k+1}}^{-\epsilon(-1)^{a_k-1}=-\eta_k} \oplus \overbrace{\chi_V^{-1}\chi_W \otimes \nu_{a_k} \otimes \nu_{b_k}}^{t_k=\frac{a_k-1}{2},-\eta_k},\ \textit{if}\ \zeta_k=-1,$$

  $$\epsilon=-\eta_k(-1)^{a_k-1},\ t_k=\frac{a_k-1}{2}$$

- $$\chi_V^{-1}\chi_W \otimes \overset{\epsilon(-1)^{a_k-1}=\eta_k}{\nu_1 \otimes \nu_{a_k-b_k+1}} \oplus \chi_V^{-1}\chi_W \otimes \overset{t_k-1,\eta_k}{\nu_{a_k} \otimes \nu_{b_k}}, \text{ if } \zeta_k = -1,$$

$$\epsilon = \eta_k(-1)^{a_k-1},\ 0 < t_k \leq \lfloor \frac{a_k-1}{2} \rfloor$$

- $$\chi_V^{-1}\chi_W \otimes \overset{-\epsilon}{\nu_1} \otimes \nu_{a_k-b_k+1} \oplus \chi_V^{-1}\chi_W \otimes \overset{t_k-1=\frac{a_i}{2}-1,-\epsilon}{\nu_{a_k} \otimes \nu_{b_k}}, \text{ if } \zeta_k = -1,$$

$$t_k = \frac{a_k}{2};\ \text{then } \eta_k = 1$$

- Now we go down in the same way; we have that $\theta_{-(2r+1)}(\pi(\psi,t,\eta))$ is in Π_{ψ_r} for all r such that $a_{k-1}+b_{k-1}+1 \le 2r+1 \le |a_k-b_k|-1$ with the parameter which has the analogous form like the parameter of $\theta_{-(|a_k-b_k|+1)}(\pi(\psi,t,\eta))$, just with $\chi_V^{-1}\chi_W \otimes \nu_1 \otimes \nu_{2r+1}$ instead of $\chi_V^{-1}\chi_W \otimes \nu_1 \otimes \nu_{a_k-b_k+1}$ (the same values of t,η)
- we repeat the same procedure until we get to the summand with index l_0 (or k_0 if l_0 does not exist).
  Let

$$\theta_{-(a_{l_0}+b_{l_0}+1)}(\pi(\psi,t,\eta)) =$$

$$\cdots \oplus \overset{t_{l_0}=0,-\eta_{l_0}}{\chi_V^{-1}\chi_W \otimes \nu_{a_{l_0}} \otimes \nu_{b_{l_0}}} \oplus \overset{\epsilon'}{\chi_V^{-1}\chi_W \otimes \nu_1 \otimes \nu_{a_{l_0}+b_{l_0}+1}} \oplus \cdots$$

  Then, if $\epsilon' \neq -\eta_{l_0}(-1)^{a_{l_0}-1}$, we stop as, for $2r+1 < a_{l_0}+b_{l_0}+1$ we have $\theta_{-(2r+1)}(\pi(\psi,t,\eta)) = 0$.

Otherwise, we are on the “going down tower” and, we have, for $b_{l_0} - a_{l_0} + 1 < 2r + 1 \leq a_{l_0} + b_{l_0} - 1$

$$\theta_{-(2r+1)}(\pi(\psi, t, \eta)) =$$

$$\cdots \oplus \overset{t_{l_0}=0,-\eta_{l_0}}{\chi_V^{-1}\chi_W \otimes \nu_{a_{l_0}} \otimes \nu_{b_{l_0}}} \oplus \overset{\epsilon'=-\eta_{l_0}(-1)^{a_{l_0}-1}}{\chi_V^{-1}\chi_W \otimes \nu_1 \otimes \nu_{2r+1}} \oplus \cdots$$

and, if $a_{l_0} \geq 2$

$$\theta_{-(b_{l_0}-a_{l_0}+1)}(\pi(\psi,t,\eta)) =$$

$$\cdots \oplus \chi_V^{-1}\chi_W \otimes \nu_1 \otimes \nu_{b_{l_0}-a_{l_0}+1}^{-\eta_{l_0}} \oplus \chi_V^{-1}\chi_W \otimes \nu_{a_{l_0}} \otimes \nu_{b_{l_0}}^{1,\eta_{l_0}} \oplus \cdots$$

and, if $a_{l_0} = 1$ we have

$$\theta_{-(b_{l_0}-a_{l_0}+1)}(\pi(\psi,t,\eta)) =$$

$$\cdots \oplus \chi_V^{-1}\chi_W \otimes \nu_1 \otimes \nu_{b_{l_0}-a_{l_0}+1}^{\epsilon'=-\eta_{l_0}(-1)^{a_{l_0}-1}} \oplus \chi_V^{-1}\chi_W \otimes \nu_{a_{l_0}} \otimes \nu_{b_{l_0}}^{0,-\eta_{l_0}} \oplus \cdots$$

- Now, on the "going down tower" we proceed "going down" and "jumping over" summands in the same way as explained above for summands before (i. e. larger) than l_0
- We stop at the first (going from the right; i.e. the largest) summand $i > k_0$ for which $\zeta_i = -1,\ t_i = 0$ and ϵ'' (which is obtained by the previous steps) is such that

$$\theta_{-(a_i+b_i+1)}(\pi(\psi, t, \eta)) =$$

$$\cdots \oplus \overset{t_i=0,-\eta_i}{\chi_V^{-1}\chi_W \otimes \nu_{a_i} \otimes \nu_{b_i}} \oplus \overset{\epsilon''=\eta_i(-1)^{a_i-1}}{\chi_V^{-1}\chi_W \otimes \nu_1 \otimes \nu_{a_i+b_i+1}} \oplus \cdots$$

  i.e. $\epsilon" \neq -\eta_i(-1)^{a_i-1}$.
- If there is no such summand, we stop at k_0 if, again, $\epsilon" \neq -\eta_{k_0}(-1)^{a_i-1}$ (here it is not necessary that $\zeta_{k_0} = -1$ to be forced to stop).

- if we do not stop at k_0, i.e. if $\epsilon'' = -\eta_{k_0}(-1)^{a_i-1}$, we use the above rules to "jump over" summands (we are in the "cuspidal part" now) by following the rules for $t_i = 0$ and $\zeta_i = 1$ or $\zeta_i = -1$, as given above. By examining the changes of characters, we see that we will stop after jumping over the summand which is our first encounter with $\zeta_i = -1$ (from above; i.e. from the right). In other words, in this case, the last theta lift in A-packet will be $\theta_{-(b_i-a_i+1)}(\pi(\psi, t, \eta))$ where i is the largest index less or equal to k_0 such that $\zeta_i = -1$.

- In this way we have completely answered the first question of Mœglin: we describe where we stop with "going down" at the theta-tower (i.e. calculating $\theta_{-(2r+1)}(\pi(\psi, t, \eta)))$ in a way that all the theta lifts are in appropriate Arthur packet ψ_r. (so that the theta lift on just one lever lower is not in appropriate A-packet). We have explicitly given the parameters of the lifts.
- As for the second question: this is quite (quite!) conjectural just now, but the evidence suggests that, actually, once the ascending in the theta tower stops, there are no lower theta lifts which belong to appropriate A-packets of the form ψ_r (the lifts themselves are non-zero if we are on the "going-down" tower, but they do not belong to the A-packets of the right form)

About proofs

- In our paper P.Bakić, M.H. Theta correspondence for p-adic dual pairs of type I, Crelle, 2021. we gave the explicit form of theta correspondence, i.e. gave the Langlands parameter of $\theta_l(\pi)$ if π is given as a Langlands quotient
- **Problem** We do not know explicit Langlands parameters of the representations in A-packets; their construction is recursive and technical, and does not give us the explicit Langlands parameter, as of yet (so no direct use of our previous results)
- We do use some features of our description, but also, we:
- use and refine some results of Mœglin about reducibility for representations in DDR A-packets
- use results of Atobe and Minguez on the derivatives for classical groups
- use some *GL*-reducibility results (e.g. Lapid and Minguez)

Thank you!