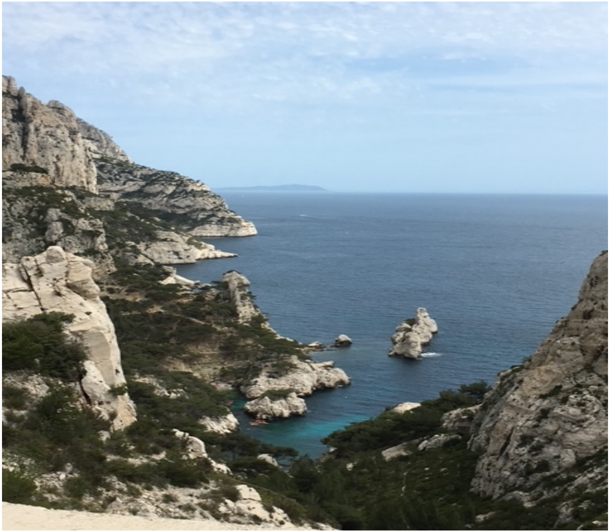
Howe to transfer Harish-Chandra characters via Weil's representation



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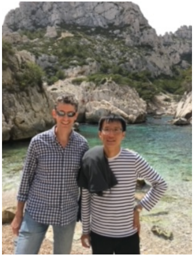

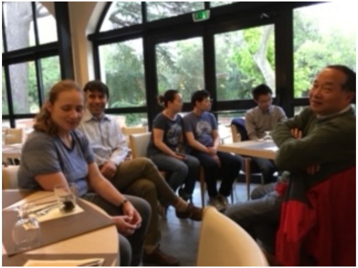

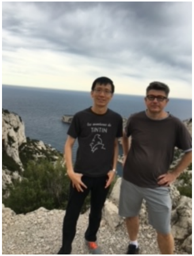

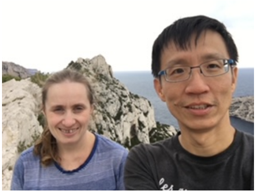

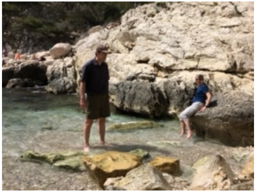

Dual Pair

- F a p-adic field.
- W symplectic vector space of dimension $2n$;
- $V = V^{\pm}$ quadratic spaces of dim $= 2m + 1$ and disc. 1.

Have dual pair

$$\mathrm{Sp}(W) \times \mathrm{O}(V) \longrightarrow \mathrm{Sp}(W \otimes V).$$

Let $\mathrm{Mp}(W)$ be the unique two-fold nonlinear cover of $\mathrm{Sp}(W)$.

Have:

$$\mathrm{Mp}(W) \times \mathrm{O}(V) \longrightarrow \mathrm{Mp}(W \otimes V).$$

Weil Representation and Theta Correspondence

For a fixed nontrivial character ψ of F, let

$$\omega_\psi = \text{Weil rep. of } \mathrm{Mp}(W) \times \mathrm{O}(V).$$

For $\pi \in \mathrm{Irr}(\mathrm{O}(V))$, define a smooth rep. of $\mathrm{Mp}(W)$ by

$$\Theta(\pi) = (\omega_{V,W,\psi} \otimes \pi^\vee)_{\mathrm{O}(V)} \quad \text{(big theta lift)}.$$

Likewise, for $\tilde{\pi} \in \mathrm{Irr}(\mathrm{Mp}(W))$, have smooth rep. $\Theta(\tilde{\pi})$ of $\mathrm{O}(V)$.

Theorem (Howe Duality)

(i) $\Theta(\pi)$ has finite length and a unique irreducible quotient $\theta(\pi)$.

(ii) If $\pi_1 \neq \pi_2$, then $\theta(\pi_1) \neq \theta(\pi_2)$ (if both nonzero).

Equal Rank Case

We shall focus on the special case $m = n$, so that $\dim V^{\pm} = \dim W + 1 = 2n + 1$.

Theorem (Local Shimura Correspondence)

The theta correspondence, together with the restriction from $O(V)$ to $SO(V)$, gives a bijection

$$\mathrm{Irr}(Mp(W) \longleftrightarrow \mathrm{Irr}(SO(V^+)) \sqcup \mathrm{Irr}(SO(V^-)).$$

Moreover, under this bijection, discrete series representations correspond, and so do tempered representations.

$$\theta : \mathrm{Irr}_{temp}(\mathrm{SO}(V^+)) \sqcup \mathrm{Irr}_{temp}(\mathrm{SO}(V^-)) \longleftrightarrow \mathrm{Irr}_{temp}(\mathrm{Mp}(W)),$$

Characters

If $\pi \in \mathrm{Irr}(G(F))$, set

$$\Theta_\pi = \text{ Harish-Chandra character of } \pi.$$

It is a conjugacy-invariant distribution on $G(F)$, which is given by a locally L^1 smooth function on the regular semisimple locus:

$$\Theta_\pi : C_c^\infty(G(F)) \twoheadrightarrow C_c^\infty(G(F))_{G(F)^\Delta} \to \mathbb{C}.$$

If π is unitary and $\{e_i\}$ is an orthonormal basis of π, then

$$\Theta_\pi(f) = \mathrm{Tr}(\pi(f)) = \sum_i \langle \pi(f)e_i, e_i \rangle.$$

If π is tempered, then Θ_π is a tempered distribution: it extends to a linear form on the Harish-Chandra-Schwarz space $C_c^\infty(G(F)) \subset \mathcal{C}(G(F)) \subset L^2(G(F))$.

The Question

Suppose $\tilde{\pi} \in \mathrm{Irr}(\mathsf{Mp}(W))$ and $\pi \in \mathrm{Irr}(\mathsf{SO}(V^{\epsilon}))$ satisfy $\tilde{\pi} = \theta(\pi)$.

Question: $\boxed{\text{How are the characters of } \pi \text{ and } \tilde{\pi} \text{ related?}}$

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- This question has been studied by T. Prezbinda when $F = \mathbb{R}$; he introduced a construction known as the Cauchy-Harish-Chandra integral, which transfers invariant eigendistributions from one group to another and conjectured that this relates the characters of representations in theta correspondence with each other.
- He verified his conjecture in the stable range. A recent paper of A. Merino verified this for discrete series in $\mathrm{U}(n) \times \mathrm{U}(n)$ case. The analytic difficulties in working with this integral is one obstacle in extending these results.

$\boxed{\text{We would like to propose a more conceptual approach.}}$

An Approach to Character Relations

We shall:

- introduce spaces of test functions on $\mathrm{Mp}(W)$ and $\mathrm{SO}(V)$;
- define a notion of transfer of test functions from one group to another;
- show that this transfer descends to a well-defined map on the level of coinvariant spaces (i.e. orbital integrals), thus inducing a transfer of invariant distributions.
- show that the transfer of Θ_π is equal to $\Theta_{\tilde{\pi}}$.
- describe the transfer in geometric terms (in terms of a moment map).

Spaces of Test Functions

$$\begin{tikzcd}[column sep=small] & \Omega^{\epsilon} = \omega_{W,V^{\epsilon}} \otimes \overline{\omega_{W,V^{\epsilon}}} \arrow[dl, "p^{\epsilon}"'] \arrow[dr, "q^{\epsilon}"] & \\ C^{\infty}(\mathrm{Mp}(W)) & & C^{\infty}(\mathrm{SO}(V^{\epsilon})) \end{tikzcd}$$

The two maps are defined by matrix coefficients:

$$p^{\epsilon}(\phi_1 \otimes \phi_2)(g) = \langle \phi_1, g\phi_2 \rangle \quad \text{and} \quad q^{\epsilon}(\phi_1 \otimes \phi_2)(h) = \langle \phi_1, h\phi_2 \rangle.$$

Set

$$\mathcal{S}^{\epsilon}(\mathrm{Mp}(W)) = \mathrm{Image}(p^{\epsilon}) \quad \text{and} \quad \mathcal{S}(\mathrm{SO}(V^{\epsilon})) = \mathrm{Image}(q^{\epsilon}).$$

These are our spaces of test functions.

Alternatively, Ω^{ϵ} can be naturally realized on $\mathcal{S}(V^{\epsilon} \otimes W)$. Then

$$p(\Phi)(g) = (g, 1) \cdot \Phi(0) \quad \text{and} \quad q(\Phi)(h) = (h, 1) \cdot \Phi(0).$$

Properties of Test Functions

Lemma

$$C_c^\infty(SO(V^\epsilon)) \subset \mathcal{S}(SO(V^\epsilon)) \subset \mathcal{C}(SO(V^\epsilon)).$$

Moreover, the map $q^\pm : \Omega^\pm \longrightarrow \mathcal{S}(SO(V^\pm)$ *induces an isomorphism*

$$(\Omega^\pm)_{Mp(W)^\Delta} \cong \mathcal{S}(SO(V^\pm)).$$

Corollary

For $\pi \in \mathrm{Irr}_{temp}(SO(V^\epsilon))$ *and* $f \in \mathcal{S}(SO(V^\epsilon))$ *the operator* $\pi(f)$ *is defined and so is its trace.*

Transfer of Test Functions

We say that

$$f^\epsilon \in \mathcal{S}(\mathrm{SO}(V^\epsilon)) \quad \text{and} \quad \tilde{f}^\epsilon \in \mathcal{S}^\epsilon(\mathrm{Mp}(W))$$

are transfer of each other if there exists $\Phi \in \Omega^\epsilon$ such that

$$p^\epsilon(\Phi) = f^\epsilon \quad \text{and} \quad q^\epsilon(\Phi) = \tilde{f}^\epsilon.$$

More generally, say that

$$f = (f^+, f^-) \in \mathcal{S}(\mathrm{SO}(V^\pm)) := \mathcal{S}(\mathrm{SO}(V^+)) \oplus \mathcal{S}(\mathrm{SO}(V^-))$$

and

$$\tilde{f} = (\tilde{f}^+, \tilde{f}^-) \in \mathcal{S}(\mathrm{Mp}(W)) := \mathcal{S}^+(\mathrm{Mp}(W)) \oplus \mathcal{S}^-(\mathrm{Mp}(W))$$

are in correspondence if the $\pm$-components correspond.
Transfers always exist, by definition.

Spaces of Orbital Integrals

Consider the composite:

$$\Omega = \oplus_\epsilon \Omega^\epsilon \to \mathcal{S}(\mathrm{Mp}(W)) \to \mathcal{S}(\mathrm{Mp}(W))_{\mathrm{Mp}(W)^\Delta}.$$

This map is $\mathrm{Mp}(W)^\Delta$-invariant and thus factors as:

$$\Omega \to \Omega_{\mathrm{Mp}(W)^\Delta} \cong \mathcal{S}(\mathrm{SO}(V^\pm)) \to \mathcal{S}(\mathrm{Mp}(W))_{\mathrm{Mp}(W)^\Delta}$$

Since the last arrow is also $\mathrm{SO}(V^\epsilon)^\Delta$-invariant, it further descends to

$$\mathcal{S}(\mathrm{SO}(V^\epsilon))_{\mathrm{SO}(V^\epsilon)^\Delta} \longrightarrow \mathcal{S}(\mathrm{Mp}(W))_{\mathrm{Mp}(W)^\Delta}.$$

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Lemma

This construction gives an isomorphism

$$\mathcal{S}(SO(V^+))_{SO(V^+)^\Delta} \oplus \mathcal{S}(SO(V^-))_{SO(V^-)^\Delta} \cong \mathcal{S}(Mp(W))_{Mp(W)^\Delta}.$$

A Character Identity

The previous lemma allows one to transfer invariant distributions between $\mathcal{S}(\mathrm{Mp}(W))$ and $\mathcal{S}(\mathrm{SO}(V^{\pm}))$.

Theorem

Suppose that $\pi \in \mathrm{Irr}_{temp}(SO(V^{\epsilon}))$, so that $\tilde{\pi} = \theta(\pi) \in \mathrm{Irr}_{temp}(Mp(W))$. Then for f and $\tilde{f}$ in correspondence,

$$\Theta_{\pi}(f) = \Theta_{\tilde{\pi}}(\tilde{f}).$$

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This theorem was first shown by Hang Xue, in the guise of an identity between two doubling zeta integrals.

$\boxed{\text{What we will discuss in the rest of this talk}}$

- a sketch proof of the character identity.
- a geometric description of the transfer of test functions.

Sketch Proof via the Plancherel Theroem

For $\Phi \in S(V^{\epsilon} \otimes W) = \Omega^{\epsilon}$, observe that

$$p(\Phi)(1) = \Phi(0) = q(\Phi)(1).$$

Since $p(\Phi) \in \mathcal{C}(\mathrm{Mp}(W))$, the Harish-Chandra-Plancherel theorem gives:

$$p(\Phi)(1) = \int_{\widehat{\mathrm{Mp}(W)}} \Theta_{\tilde{\pi}}(p(\Phi))\, d\mu_{\mathrm{Mp}(W)}(\tilde{\pi}).$$

Likewise,

$$q(\Phi)(1) = \int_{\widehat{\mathrm{SO}(V)}} \Theta_{\pi}(q(\Phi))\, d\mu_{\mathrm{SO}(V)}(\pi).$$

So we get the equality of both RHS's.

We have shown:

$$\int_{\widehat{\mathrm{Mp}(W)}} \Theta_{\tilde{\pi}}(p(\Phi))\, d\mu_{\mathrm{Mp}(W)}(\tilde{\pi}) = \int_{\widehat{\mathrm{SO}(V)}} \Theta_{\pi}(q(\Phi))\, d\mu_{\mathrm{SO}(V)}(\pi).$$

Under the local Shimura correspondence

$$\theta : \mathrm{Irr}_{temp}(\mathrm{SO}(V^+)) \cup \mathrm{Irr}_{temp}(\mathrm{SO}(V^-)) \longleftrightarrow \mathrm{Irr}_{temp}(\mathrm{Mp}(W)),$$

one has (by G.-Ichino)

$$\theta_*(d\mu_{\mathrm{SO}(V^+)}) + \theta_*(d\mu_{\mathrm{SO}(V^-)}) = d\mu_{\mathrm{Mp}(W)}$$

This gives

$$\int_{\widehat{\mathrm{SO}(V)}} \Theta_{\theta(\pi)}(p(\Phi))\, d\mu_{\mathrm{SO}(V)}(\pi) = \int_{\widehat{\mathrm{SO}(V)}} \Theta_{\pi}(q(\Phi))\, d\mu_{\mathrm{SO}(V)}(\pi).$$

One can peel off the integrals on both sides using a Bernstein center argument.

The Formal Setup

Start with a nonzero $G \times H$-equivariant map

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Have commutative diagram:

$$\begin{tikzcd}
 & & \Omega = \omega \otimes \overline{\omega} \arrow[dll, "p"'] \arrow[d, "\theta\otimes\bar{\theta}"'] \arrow[drr, "q"] & & \\
\mathcal{S}(G) := \Omega_{H^\Delta} \arrow[d, "p_\pi^\theta"'] & & \mathrm{End}_f(\pi) \otimes \mathrm{End}_f(\sigma) \arrow[dll, "Tr"] \arrow[drr, "Tr"'] & & \Omega_{G^\Delta} =: \mathcal{S}(H) \arrow[d, "q_\sigma^\theta"] \\
\mathrm{End}_f(\pi) \arrow[drr, "Tr"'] & & & & \mathrm{End}_f(\sigma) \arrow[dll, "Tr"] \\
 & & \mathbb{C} & &
\end{tikzcd}$$

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\mathrm{End}_f(\pi) \arrow[drr, "Tr"] & & & & \mathrm{End}_f(\sigma) \arrow[dll, "Tr"'] \\
 & & \mathbb{C} & &
\end{tikzcd}$$

To give content:

- Can $\mathcal{S}(G)$ really be realised as a space of functions on G?
- If so, is p_π the integrated version of π?

Moment Map

The moment map is a double fibration

$$
\begin{tikzcd}
 & V \otimes W \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\
\mathfrak{sp}(W)^* \cong \mathrm{Sym}^2 W^* & & \mathfrak{so}(V)^* \cong \wedge^2 V^*
\end{tikzcd}
$$

The maps are given by:

$$p'(T) = T \circ T^* \quad \text{and} \quad q'(T) = T^* \circ T.$$

- The map p' is $\mathrm{Sp}(W)$-equivariant and $\mathrm{SO}(V)$-invariant, whereas q' is $\mathrm{SO}(V)$-equivariant and $\mathrm{Sp}(W)$-invariant.
- It induces a correspondence of orbits between $\mathfrak{so}(V)$ and $\mathfrak{sp}(W)$, giving a bijection

$$\mathfrak{so}(V)^{\heartsuit}//\mathrm{SO}(V) \longleftrightarrow \mathfrak{sp}(W)^{\heartsuit}//\mathrm{Sp}(W),$$

where $\mathfrak{sp}(W)^{\heartsuit}$ correspond to maximally nondegenerate maps.

Moment Map Correspondence

The moment map diagram

$$\begin{tikzcd} & V \otimes W \arrow[dl, "p^{mom}"'] \arrow[dr, "q^{mom}"] & \\ \mathfrak{sp}(W) & & \mathfrak{so}(V) \end{tikzcd}$$

induces by integration along the fibers:

$$\begin{tikzcd} & \mathcal{S}(V \otimes W) \arrow[dl, "p_*^{mom}"'] \arrow[dr, "q_*^{mom}"] & \\ \mathcal{S}(\mathfrak{sp}(W)^\heartsuit) & & \mathcal{S}(\mathfrak{so}(V)^\heartsuit) \end{tikzcd}$$

This defines a "moment map correspondence" of the two spaces of test functions.

Cayley Transform

Lemma

There is an isomorphism of $Mp(W)^{\Delta}$-modules

$$\mathcal{S}(Mp(W)) \longrightarrow \mathcal{S}(\mathfrak{sp}(W)^{\heartsuit})$$

defined by:

$$j(f)(x) = f(c(x)) \cdot |\det(1 - c(x))|^{\frac{\dim V}{2}}.$$

where

$$c : \mathfrak{sp}(W) \longrightarrow Mp(W)$$

which is the "birational map" given by the Cayley transform

$$c(x) = (x - 1)(x + 1)^{-1}$$

(when projected to $Sp(W)$).

Character of Weil representation

The factor $|\det(1-c(x))|^{\frac{\dim V}{2}}$ which appears in the previous lemma can be interpreted in terms of the character of the Weil representation. One has the following result of Teruji Thomas:

Theorem

As a generalized function on $Mp(V \otimes W)$, the character of the Weil representation $\omega_{V,W,\psi}$ is given by

$$\mathrm{Tr}(\omega_{V,W,\psi}(g)) = \gamma_\psi(g) \cdot |\det_{V\otimes W}(g-1)|^{-1/2}.$$

For $g \in \mathrm{Mp}(W)$, one has

$$\mathrm{Tr}(\omega_{V,W,\psi}(g \otimes 1_V)) = \gamma_\psi(g) \cdot |\det_W(g-1)|^{-\dim V/2}.$$

So

$$j(f)(x) = f(c(x)) \cdot \mathrm{Tr}(\omega_{V,W,\psi}(c(x) \otimes 1_V))^{-1}.$$

Transfer and Moment Map

Proposition

$$j_W \circ p = \mathcal{F}^{\heartsuit} \circ p_*^{mom} \circ \mathcal{F}_{V\otimes W}.$$

where $\mathcal{F}_{V\otimes W}$ is the Fourier transform on $V \otimes W$ and

$$\mathcal{F}^{\heartsuit} : \mathcal{S}(\mathfrak{sp}(W)^{\heartsuit}) \longrightarrow \mathcal{S}(\mathfrak{sp}(W))$$

is the Fourier transform (of distributions) on $\mathfrak{sp}(W)$.

So have commutative diagram:

$$\begin{tikzcd}
\mathcal{S}(V \otimes W) \arrow[r, "\mathcal{F}_{V\otimes W}"] \arrow[d, "p"'] & \mathcal{S}(V \otimes W) \arrow[r, "p_*^{mom}"] & \mathcal{S}(\mathfrak{sp}(W)^{\heartsuit}) \arrow[d, "\mathcal{F}^{\heartsuit}"] \\
\mathcal{S}(\mathrm{Mp}(W)) \arrow[r, "j_W"'] & \mathcal{S}(\mathfrak{sp}(W)) \arrow[r, equal] & \mathcal{S}(\mathfrak{sp}(W))
\end{tikzcd}$$

Geometric Description of Transfer

Here is our geometric description of the transfer of test functions:

- given $\tilde{f} \in \mathcal{S}(\mathrm{Mp}(W))$ and $f \in \mathcal{S}(\mathrm{SO}(V))$, we consider

$$j_W(\tilde{f}) \in \mathcal{S}(\mathfrak{sp}(W)) \quad \text{and} \quad j_V(f) \in \mathcal{S}(\mathfrak{so}(V)).$$

- Then $\tilde{f}$ and f correspond if the Fourier transforms $\mathcal{F}^{\heartsuit}(j_W(\tilde{f}))$ and $\mathcal{F}^{\heartsuit}(j_V(f))$ correspond under the moment map correspondence.
- In that case, $\mathcal{F}^{\heartsuit}(j_W(\tilde{f}))$ and $\mathcal{F}^{\heartsuit}(j_V(f))$ have equal orbital integrals under the bijection of nondegenerate orbits induced by the moment map.

Periods and Theta Correspondence

The techniques for deriving the main character identity can be applied in the setting of the relative Langlands program to give relative character identities.

What we did was to relate the following symmetric spaces via Θ-correspondence:

$$\mathrm{Sp}(W) \times \mathrm{Sp}(W)/\mathrm{Sp}(W)^{\Delta} \longleftrightarrow \mathrm{SO}(V) \times \mathrm{SO}(V)/\mathrm{SO}(V)^{\Delta}$$

But Θ-correspondence can relate other periods of a dual pair $G \times H$.

Suppose a period $\mathcal{P}$ on G is related to a period $\mathcal{Q}$ on H, i.e. for $\pi \in \mathrm{Irr}(G)$, i.e.

$$\Theta(\pi) \text{ is } \mathcal{Q}\text{-distingusihed} \Longleftrightarrow \pi \text{ is } \mathcal{P}\text{-distinguished.}$$

Question: can the relative characters $\mathrm{Tr}_{\mathcal{P}}(\pi)$ and $\mathrm{Tr}_{\mathcal{Q}}(\Theta(\pi))$ be related?

Examples

- Rank 1 examples:

$$\mathrm{O}_{n-1}\backslash \mathrm{O}_n, \quad \mathrm{SL}_3\backslash G_2, \quad \mathrm{Spin}_9\backslash F_4.$$

  These are related to $(N,\psi)\backslash \mathrm{SL}_2$, with dual pairs

$$\mathrm{SL}_2 \times \mathrm{O}_n \subset \mathrm{Sp}_{2n}, \quad \mathrm{Mp}_2 \times G_2 \subset \mathrm{Sp}_{14} \quad \mathrm{PGL}_2 \times F_4 \subset E_7$$

- Rank 2 examples:

$$\mathrm{SL}_3(F)\backslash \mathrm{SL}_3(E), \quad \mathrm{Sp}_6\backslash \mathrm{SL}_6, \quad F_4\backslash E_6.$$

  These are related to $(N,\psi)\backslash \mathrm{SL}_3$, via $\mathrm{SL}_3 \times H$.

- Another rank 2 example (Wan's thesis)

$$\mathrm{U}_2\backslash \mathrm{SO}_5 \longleftrightarrow (N,\psi)\backslash \mathrm{PGL}_2 \times T\backslash \mathrm{PGL}_2,$$

  using dual pair $\mathrm{PGSp}_4 \times \mathrm{PGSO}_4$.

- Rank 3 examples:

$$G_2\backslash Spin_8 \longleftrightarrow ((N,\psi)\backslash \mathrm{SL}_2)^3 \quad \text{via} \quad \mathrm{SL}_2^3 \times \mathrm{Spin}_8 \subset E_7.$$

Spherical Harmonics

Let V be a split quadratic space with even dimension ≥ 4. Let $v_0 \in V$ be a unit vector, so that

$$V = Fv_0 \oplus U \quad \text{and} \quad \mathrm{O}(U) \subset \mathrm{O}(V).$$

The symmetric space $X = \mathrm{O}(U)\backslash\mathrm{O}(V)$ is related to the Whittaker space $(N, \psi)\backslash\mathrm{SL}_2$:

- If $\sigma \in \mathrm{Irr}_{temp}(\mathrm{SL}_2)$ and $\pi = \Theta_\psi(\sigma)$, then
$$\mathrm{Hom}_{\mathrm{O}(U)}(\pi, \mathbb{C}) \cong \mathrm{Hom}_N(\sigma, \psi),$$
where $N =$ maximal unipotent of SL_2.
- If
$$L^2(N, \psi\backslash\mathrm{SL}_2) \cong \int_{\widehat{\mathrm{SL}_2}} \sigma^\vee_{N,\psi} \otimes \sigma \, d\mu_{\mathrm{SL}_2}(\sigma)$$
then
$$L^2(\mathrm{O}(U)\backslash\mathrm{O}(V)) \cong \int_{\widehat{\mathrm{SL}_2}} \sigma^\vee_{N,\psi} \otimes \Theta_\psi(\sigma) \, d\mu_{\mathrm{SL}_2}(\sigma).$$

Bernstein's interpretation

According to Bernstein, to give a direct integral decomposition as above is equivalent to one of the following equivalent data:

- a measurable family of equivariant projections

  $$\alpha_\sigma : C_c^\infty(N, \psi \backslash \mathsf{SL}_2) \longrightarrow \sigma,$$

  giving by duality

  $$\overline{\beta}_\sigma : \overline{\sigma} \longrightarrow C^\infty(N, \psi \backslash \mathsf{SL}_2),$$

  so that

  $$f(g) = \int_{\widehat{\mathsf{SL}_2}} \beta_\sigma \circ \alpha_\alpha(f)(g)\, d\mu_\sigma.$$

- a decomposition of the inner product on $C_c^\infty(N, \psi \backslash \mathsf{SL}_2)$:

  $$\langle f_1, f_2 \rangle_{N \backslash SL_2} = \int_{\widehat{\mathsf{SL}_2}} J_\sigma(f_1, f_2)\, d\mu_\sigma.$$

  where J_σ is a positive-semidefinite inner product which factors through $\sigma \otimes \overline{\sigma}$.

Relative Characters

The two notions above are related by:

$$J_\sigma(f_1, f_2) = \langle \alpha_\sigma(f_1), \alpha_\sigma(f_2) \rangle_\sigma.$$

By a relative character, one means one of the following:

- the positive semidefinite inner product J_σ:

$$J_\sigma(f_1, f_2) = \langle f_1, \beta_\sigma \circ \alpha_\sigma(f_2) \rangle_{N \backslash \mathrm{SL}_2}.$$

- the (N, ψ)-invariant linear form

$$R_\sigma : C_c^\infty(N, \psi \backslash \mathrm{SL}_2) \longrightarrow \mathbb{C}$$

  defined by

$$R_\sigma(f) = \beta_\sigma(\alpha_\sigma(f))(1).$$

From the spectral decomposition of $L^2(X) = L^2(\mathrm{O}(U) \backslash \mathrm{O}(V))$, one also has $\alpha_{\Theta(\sigma)}$ and $\beta_{\Theta(\sigma)}$, and hence $R_{\Theta(\sigma)}$.

Transfer of Test Functions

Question: Are the relative characters R_σ for $(N,\psi)\backslash \mathrm{SL}_2$ and $R_{\Theta(\sigma)}$ for X related?

$$\begin{array}{ccc} \Omega = C_c^\infty(V) & \xrightarrow{q} & C_c^\infty(X) \\ {\scriptstyle p}\big\downarrow & & \\ C^\infty(N,\psi\backslash \mathrm{SL}_2) & & \end{array}$$

where

$$q(\Phi)(h) = h\cdot\Phi(v_0) = \Phi(h^{-1}\cdot v_0)$$

is the restriction map to $X = \mathrm{O}(V)\cdot v_0$ and is surjective since X is a closed subset of V, and

$$p(\Phi)(g) = (g\cdot\Phi)(v_0).$$

Denote the image of p by $\mathcal{S}(N,\psi\backslash SL_2)$. These are our spaces of test functions.

Properties

$$\begin{array}{ccc} \Omega = C_c^\infty(V) & \xrightarrow{q} & \mathcal{S}(X) \\ {\scriptstyle p}\big\downarrow & & \\ \mathcal{S}(N,\psi\backslash \mathrm{SL}_2) & & \end{array}$$

- p is SL_2-equivairant and $\mathrm{O}(U)$-invariant whereas q is $\mathrm{O}(V)$-equivairant and (N,ψ)-invariant.
- q induces an isomorphism $\Omega_{N,\psi} \cong \mathcal{S}(X)$.
- the map p induces

$$\mathcal{S}(X) \cong \Omega_{N,\psi} \twoheadrightarrow \mathcal{S}(N,\psi\backslash \mathrm{SL}_2)_{N,\psi}.$$

- $C_c^\infty(N,\psi\backslash \mathrm{SL}_2) \subset \mathcal{S}(N,\psi\backslash \mathrm{SL}_2) \subset \mathcal{C}(N,\psi\backslash \mathrm{SL}_2)$.

Such transfer of test functions in the setting of rank 1 spherical varieties was defined in a recent series of papers by Sakellaridis.

Relative Character Identities

Now the relative character R_{α_σ} for $(N,\psi)\backslash \mathrm{SL}_2$ extends to the Whittaker-Schwarz space $\mathcal{C}(N,\psi\backslash \mathrm{SL}_2)$. Moreover, the map

$$\mathcal{S}(X) \cong \Omega_{N,\psi} \twoheadrightarrow \mathcal{S}(N,\psi\backslash \mathrm{SL}_2)_{N,\psi}$$

descends to

$$t : \mathcal{S}(X)_{\mathrm{O}(U)} \longrightarrow \mathcal{S}(N,\psi\backslash \mathrm{SL}_2)_{N,\psi}.$$

Thus we can pullback R_σ to get a linear form on $S(X)_{\mathrm{O}(U)}$.

Theorem

$$R_\sigma \circ t = R_{\Theta(\sigma)}$$

Transfer as an Integral Transform

In this particular example, we can actually write down an explicit formula for the transfer map

$$t : \mathcal{S}(X)_{\mathrm{O}(U)} \longrightarrow \mathcal{S}(N, \psi \backslash \mathrm{SL}_2)_{N,\psi}.$$

Have:

$$X^{\heartsuit}/\mathrm{O}(U) \longleftrightarrow F \smallsetminus \{\pm 1\}, \quad \text{via} \quad v \mapsto \langle v, v_0 \rangle,$$

and

$$N \backslash (\mathrm{SL}_2 \smallsetminus T)/N \longleftrightarrow wT \cong F^{\times}.$$

One has:

$$t(\phi)(wa) = |a|^{\frac{1}{2}\dim V} \cdot \int_{F \backslash \{\pm 1\}} \phi(\xi) \cdot \psi(a^{-1}\xi)\, d\xi.$$

This agrees with the formula given in Sakellaridis' papers.

Moment Map Interpretation

$$
\begin{tikzcd}
 & V \otimes W \arrow[dl, "p^{mom}"'] \arrow[dr, "q^{mom}"] & \\
\mathfrak{sp}(W) \arrow[d, "b"'] & & \mathfrak{so}(V)^* \arrow[d, "d"] \\
\mathfrak{n}^* & & \mathfrak{so}(U)^*
\end{tikzcd}
$$

The moment map sets up an identification

$$e^* + \mathrm{Ker}(ad(f)) = b^{-1}(e^*)//N \longleftrightarrow d^{-1}(0)//\mathrm{SO}(U) = U//\mathrm{SO}(U)$$

which is an infinitesimal version of

$$N \backslash \mathrm{SL}_2 / N \longleftrightarrow \mathrm{SO}(U) \backslash \mathrm{SO}(V) / \mathrm{SO}(U).$$

The transfer map is a transform on suitable spaces of functions on these spaces.

THANK YOU FOR YOUR ATTENTION!