Ext-vanishing phenomenon in branching laws

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Relative aspects of the Langlands program
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Classical results of Ext-vanishing

Let G be a reductive group over non-Archimedean field F (assume $Z(G)$ finite in the meanwhile)

1. Cuspidal repns π: projective (and injective) in $\mathrm{Rep}^\infty(G)$ i.e. for $i \geq 1$, any $\pi' \in \mathrm{Rep}^\infty(G)$,

$$\mathrm{Ext}^i_G(\pi, \pi') = 0$$

(Bernstein, projectivity $\Rightarrow$ cuspidal by Adler-Roche, etc)

2. Sq. integrable repns: projective in $\mathrm{Rep}^{\infty,\mathrm{temp}}(G)$ (Silberger, Meyer, Opdam-Solleveld, etc)
3. Standard repns: $\pi = I(P, \sigma, \nu)$ and $\pi' = I(P', \sigma', \nu')$. If $(P, \nu) \neq (P', \nu')$, then for all i,

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Branching law

Let $H \subset G$ be closed and reductive. One guiding principle is to extend previous results to branching laws! e.g.

$$\pi \in \mathrm{Irr}(G) \text{ cuspidal} \Rightarrow \pi|_H \text{ projective}$$

For the remaining of the talk, we are mainly interested in (G, H) be one of the following Rankin-Selberg/Gan-Gross-Prasad pairs:

$$(\mathrm{GL}_{n+1}, \mathrm{GL}_n), \quad (\mathrm{SO}_{n+1}, \mathrm{SO}_n), \quad (\mathrm{U}_{n+1}, \mathrm{U}_n)$$

In a series of papers (some jt. with G. Savin), we roughly prove:

Theorem (C.-Savin, C.)

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Cuspidal case

We view GL_n as subgroup of GL_{n+1} via the embedding:

$$g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$$

Remark

GL_n does not have finite center. For cuspidal π of GL_{n+1}, we still have:

$$\pi|_{\mathrm{GL}_n} \cong \mathrm{ind}_{U_n}^{\mathrm{GL}_n} \psi \text{ is projective}$$

In general, when $\pi|_{\mathrm{GL}_n}$ is projective? With Savin, we show that for sq. int. repn. π of GL_{n+1}, $\pi|_{\mathrm{GL}_n}$ is also projective.

Relatively projective representations

Theorem (C.-Savin 21, C. 21)

Let $\pi \in \mathrm{Irr}(\mathrm{GL}_{n+1})$. Then the following conditions are equivalent:

1. $\pi|_{\mathrm{GL}_n}$ *is projective*
2. π *is* GL_{n+1}*-generic and any irreducible quotient of* $\pi|_{\mathrm{GL}_n}$ *is generic*
3. $\pi|_{\mathrm{GL}_n} \cong \mathrm{ind}_{U_n}^{\mathrm{GL}_n}\psi$ *(Gelfand-Graev representation).*

The proof uses Hecke algebras established by Bushnell-Kutzko in GL case.

Theorem (C. 21)

Let $\pi \in \mathrm{Irr}(\mathrm{GL}_{n+1})$. Then $\pi|_{\mathrm{GL}_n}$ is projective if and only if either

1. π *is essentially square-integrable; or*
2. $\pi \cong \sigma_1 \times \sigma_2$ *for some cuspidal* $\mathrm{GL}_{(n+1)/2}$*-repns* σ_i.

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Tempered representations

A conjecture of D. Prasad (or some variants) is:

Conjecture (Prasad 18)

Let (G, H) be a GGP pair. Let π_1 and π_2 be tempered repns of G and H respectively. Then

$$\mathrm{Ext}^i_H(\pi_1|_H, \pi_2) = 0$$

for $i \geq 1$.

Theorem (C.-Savin 21)

Let π_1 and π_2 be generic repns of GL_{n+1} and GL_n respectively. Then, for $i \geq 1$,

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Ext-vanishing for standard repns

Theorem (C. 2021^+)

Let π_1 and π_2 be standard repns of GL_{n+1} and GL_n respectively. Then,

$$\mathrm{Hom}_{\mathrm{GL}_n}(\pi_1|_{\mathrm{GL}_n}, \pi_2^\vee) \cong \mathbb{C},$$

and for $i \geq 1$,

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- The Hom part improves the multiplicity one theorem of Aizenbud-Gourevitch-Rallis-Schiffmann (and Sun-Zhu).
- The Ext part improves the generic Ext-vanishing conjecture of Prasad (proved by C.-Savin).
- The Hom part for (SO(4), SO(3)) is shown by D. Loeffler.

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- Jacquet–Piatetski-Shapiro–Shalika established an element in the Hom via Rankin-Selberg integral
- Also show for Archimedean F for Hom part of FJ case
- The proof is based on the use of Left-Right BZ derivatives!
- Results also hold for other Bessel models and Fourier-Jacobi models using GGP type reduction (C.20+)

Euler-Poincaré pairing

In general, Ext-groups are difficult to compute. Instead, one may consider the Euler-Poincaré pairing: for $\pi_1 \in \mathrm{Alg}_f(G)$ and $\pi_2 \in \mathrm{Alg}_f(H)$:

$$\mathrm{EP}_H(\pi_1, \pi_2) = \sum (-1)^i \dim \mathrm{Ext}^i_H(\pi_1|_H, \pi_2)$$

Theorem

When $G = \mathrm{GL}_{n+1}$ and $H = \mathrm{GL}_n$,

1. *(D. Prasad, Aizenbud-Sayag)* $\dim \mathrm{Ext}^i_G(\pi_1, \pi_2) < \infty$*;*
2. *(D. Prasad)*

$$\mathrm{EP}_H(\pi_1, \pi_2) = \dim \mathrm{Wh}(\pi_1) \dim \mathrm{Wh}(\pi_2).$$

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A consequence

- Let $\pi \in \mathrm{Irr}(\mathrm{GL}_n)$ be non-generic. One write in Grothendieck group:

$$[\pi] = \sum_{\pi' \in \mathrm{Irr}(\mathrm{GL}_n)} m_{\pi,\pi'}[I(\pi')],$$

where $m_{\pi,\pi'} \in \mathbb{Z}$ and $I(\pi')$ is the standard repn. with quotient π'.

- The formula of Prasad gives that $\mathrm{EP}(\pi, \tau) = 0$ for any $\tau \in \mathrm{Irr}(\mathrm{GL}_{n-1})$.

If we take τ to be generic, we get:

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Theorem (C.)

Let $\pi \in \mathrm{Irr}(\mathrm{GL}_{n+1})$ *be not a character. Let* $\pi' \in \mathrm{Irr}(G_n)$*. Then*

$$\mathrm{Ext}^d_{\mathrm{GL}_n}(\pi, \pi') = 0,$$

where d *is the cohomological dimension of the* G_n*-Bernstein block of* π'*.*

Proof: Apply cohomological duality of Nori-Prasad,

$$\mathrm{Ext}^d_{\mathrm{GL}_n}(\pi, \pi') \cong \mathrm{Hom}_{\mathrm{GL}_n}(\mathbb{D}(\pi'), \pi)^\vee = 0,$$

where $\mathbb{D}$ is the Aubert-Schneider-Stuhler-Zelevinsky involute. The proof of last equality is based on a use of Left-Right Bernstein-Zelevinsky filtration and proved in C.21!

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Thank you!