Quasi logistic distributions and Gaussian scale mixing

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Program

- The Gaussian scale mixing, the logistic and Kolmogorov-Smirnov distributions, and the anonymous physicist question
- The quasi logistic distributions
- The quasi Kolmogorov Smirnov distributions
- More about Gaussian scale mixing in several dimensions
- The L^2 approximation

Foreword

This is an elementary lecture on the unfashoniable distribution theory: but you can pick exercises from it for your undergraduate classes....

Gaussian scale mixing

If $Z \sim N(0, I_n)$ is independent of the positive definite random matrix V then
the distribution of $X = \sqrt{V}Z$ is called a Gaussian scaled mixing distribution.

Example: $n = 1$ and $V \sim \frac{1}{2}\delta_1 + \frac{1}{2}\delta_4$

But the law of V can be continuous

Example $n = 1$
$V \sim e^{-v/2}1_{(0,\infty)}(v)dv/2$ is an exponential distribution of mean 2 independent of $Z \sim N(0,1)$ implies that $X = \sqrt{V}Z$ has the bilateral density $e^{-|x|}/2$. Indeed

$$\mathbb{E}(e^{itX}) = \mathbb{E}(\mathbb{E}(e^{it\sqrt{V}Z}|V)) = \mathbb{E}(e^{-t^2V^2/2}) = \frac{1}{1+t^2}.$$

After all, this is just a multiplicative deconvolution ?

For $n = 1$

$$\boxed{2\log|X| = \log Z^2 + \log V}$$

which means that if we wonder if the law of X is a Gaussian scale mixing we have just to check whether or not its Mellin transform $M_{X^2}(s)$ divided by the Mellin transform $2^s\Gamma(1+\frac{s}{2})$ of Z^2 is the Mellin transform $M_V(s)$ of some random variable V?

The beautiful example of Edwards-Mallows- Monahan-Stefanski

These statisticians have observed in 1973 and 1990 that if

$$\boxed{\Pr(X < x) = \frac{1}{1+e^{-x}}}$$

has the **logistic distribution** then this law is a Gaussian mixing, with $Y = \sqrt{V}$ having the **Kolmogorov-Smirnov distribution**

$$\boxed{\Pr(Y < y) = 2\sum_{n=1}^{\infty}(-1)^{n-1}e^{-2n^2y^2}}$$

(Think of this distribution function of V: the fact that it is increasing is not obvious!)

The anonymous physicist

On a mathematical site he has asked for the probability measure $\mu_{a,b}(dv)$ such that for $0 < a < b$

$$\boxed{\int_0^\infty e^{-sv}\mu_{a,b}dv = \frac{b\sinh a\sqrt{s}}{a\sinh b\sqrt{s}}}$$

Since Kolmolmogorov Smirnov is more or less $\mu_{0,b}$ what about a little generalization on Edwards- Mallows- Monahan- Stefanski? And a little generalization of the logistic distribution?

The quasi logistic distributions

They are densities proportional to

$$\frac{1}{2(\cosh x + \theta)} = \frac{e^x}{e^{2x} + 2\theta e^x + 1}$$

with $\theta > -1$. The case $\theta = 1$ is the logistic one. The shape of the curve ressembles to the normal curve, but the asymptotic is rather $e^{-|x|}$ rather than $e^{-x^2/2}$. For our purposes of the day, we concentrate to the case

$$-1 < \theta = \cos a < 1$$

with $0 < a < \pi$. The next theorem lists their properties (however the case $\theta > 1$ remains interesting in itself).

Quasi logistic laws of parameter $\theta = \cos a$: properties

Theorem 1: Let $0 < a < \pi$ and $s \in (-1, 1)$.

1. We have
$$\int_{-\infty}^{\infty} \frac{e^{sx}dx}{2(\cosh x + \cos a)} = \frac{\pi}{\sin \pi s} \times \frac{\sin as}{\sin a}. \qquad (1)$$

2. In particular if
$$X \sim \frac{\sin a}{a} \frac{dx}{2(\cosh x + \cos a)}$$
has the quasi logistic distribution of parameter $\theta = \cos a$ then for real t we have
$$\mathbb{E}(e^{sX}) = \frac{\pi s}{\sin \pi s} \times \frac{\sin as}{as}, \quad \mathbb{E}(e^{itx}) = \frac{\pi t}{\sinh \pi t} \times \frac{\sinh at}{at}$$

Other properties of the QL laws 1

1. The variance of $X \sim -X$ and the fourth moment are

$$\mathbb{E}(X^2) = \frac{1}{3}(\pi^2 - a^2), \quad \mathbb{E}(X^4) = \frac{1}{15}(\pi^2 - a^2)(7\pi^2 - 3a^2).$$

2. The distribution function of X is

$$F(x) = \Pr(X < x) = 1 - \frac{1}{a} \operatorname{Arc\,cotan} \frac{e^x + \cos a}{\sin a}$$

and the quantile function $Q(p)$ defined for $p \in (0, 1)$ by $F(Q(p)) = p$ is equal to

$$Q(p) = \log \frac{\sin pa}{\sin(1 - p)a}.$$

Other properties of the QL laws 2

X is infinitely divisible. In particular its Lévy measure is

$$\nu(dx) = \frac{e^{-|x|/a} - e^{-|x|/\pi}}{(1 - e^{-|x|/\pi})(1 - e^{-|x|/a})} \times \frac{dx}{|x|}$$

with $\int_R \min(1, |x|)\nu(dx) = \infty$.

Other properties of the QL laws 3

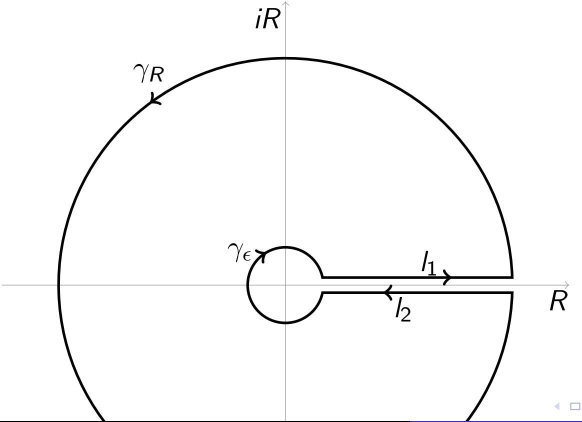
1. If $(\epsilon_n)_n \geq 1$ are Bernoulli iid rv such that $\Pr(\epsilon_n) = a^2/\pi^2$ and if $(Y_n)_n \geq 1$ are iid rv with bilateral exponential density $e^{-|y|}/2$ then
$$X \sim \sum_{n=1}^{\infty} \epsilon_n \frac{Y_n}{n}.$$
2. The Mellin transform of $|X|$ is for $s > 0$

$$\boxed{\mathbb{E}(|X|^s) = 2\Gamma(1+s)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin na}{na} \times \frac{1}{n^s}}$$

Comments about the Laplace transform

$$\int_{-\infty}^{\infty} \frac{e^{sx}dx}{2(\cosh x + \cos a)} = \int_0^{\infty} \frac{z^s dz}{z^2 + 2\cos az + 1} = \frac{\pi s}{\sin \pi s} \times \frac{\sin as}{as}$$

is not so easy, the simplest proof uses the residues calculus along the contour



Comments about the factorization 1

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right), \quad \frac{\sinh \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2}\right). \tag{2}$$

For $0 < a < b$ the second formula of (2) one leads to:

$$\frac{b \sinh \pi a t}{a \sinh \pi b t} = \prod_{n=1}^{\infty} \left(\frac{1 + \frac{a^2 t^2}{n^2}}{1 + \frac{b^2 t^2}{n^2}} \right) \tag{3}$$

Comments about the factorization 2

Let us consider the simple identity for $0 < a < b$:

$$\frac{1+a^2t^2}{1+b^2t^2} = \frac{a^2}{b^2} + (1-\frac{a^2}{b^2})\frac{1}{1+b^2t^2} \qquad (4)$$

If $Y \sim e^{-|y|}dy/2$ is a bilateral exponential random variable, we have $\mathbb{E}(e^{itY}) = 1/(1+t^2)$. If ϵ is a Bernoulli random variable such that

$$\Pr(\epsilon = 0) = 1 - \Pr(\epsilon = 1) = a^2/b^2$$

and if ϵ and Y are independent, then $\mathbb{E}(e^{itb\epsilon Y}) = (1+a^2t^2)/(1+b^2t^2)$. From this observation and from (3) we get that if $(\epsilon_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ are independent with $\epsilon_n \sim \epsilon$ and $Y_n \sim Y$ we have that

$$X = b\sum_{n=1}^{\infty} \epsilon_n \frac{Y_n}{n}$$

satisfies $\mathbb{E}(e^{itX}) = \frac{b \sinh at}{a \sinh bt}$.

Comments about the Mellin transform

If we assume that $s > 0$ we have

$$\begin{aligned}
\mathbb{E}(|X|^s) &= \frac{\sin a}{a}\int_0^\infty \frac{x^s}{\cosh x + \cos a}dx \\
&= \frac{2\sin a}{a}\int_0^\infty \frac{x^s e^{-x}}{1+2e^{-x}\cos a + e^{-2x}}dx \\
&= \frac{1}{ia}\int_0^\infty x^s\left(\frac{1}{1+e^{-x-ia}} - \frac{1}{1+e^{-x+ia}}\right)dx \\
&= \frac{1}{ia}\sum_{n=1}^\infty (-1)^{n-1}(e^{ina}-e^{-ina})\int_0^\infty e^{-nx}x^s dx
\end{aligned}$$

.

$$\mathbb{E}(|X|^s) = 2\Gamma(1+s)\sum_{n=1}^\infty (-1)^{n-1}\frac{\sin na}{na}\times\frac{1}{n^s}$$

Now the quasi Kolmogorov Smirnov laws

Theorem 2. Given $0 < a < b$, we denote $q = a/b$. There exists a probability $\mu_{a,b}(dv)$ on $(0, \infty)$ such that

$$\int_0^\infty e^{-sv}\mu_{a,b}(dv) = \frac{b\sinh(a\sqrt{s})}{a\sinh(b\sqrt{s})}. \tag{5}$$

More specifically
If $(\epsilon_n)_{n=1}^\infty$ and $(W_n)_{n=1}^\infty$ are Bernoulli and exponential independent random variables:

$$\Pr(\epsilon_n = 0) = 1 - \Pr(\epsilon_n = 1) = q^2, \quad W_n \sim e^{-w}1_{(0,\infty)}(w)dw$$

we denote $V \sim \sum_{n=1}^\infty \epsilon_n \frac{W_n}{n^2}$. Then $\frac{\pi^2}{b^2}V \sim \mu_{a,b}$ and $V \sim \mu_{\pi q,\pi}$.

The density of the quasi Kolmogorov Smirnov laws

The density of V is

$$\boxed{g(v) = \frac{2}{\pi q} \sum_{n=1}^{\infty} (-1)^{n-1} \sin(n\pi q) \times n e^{-n^2 v}}$$

In particular

$$\boxed{\mathbb{E}((\sqrt{V})^s) = 2\Gamma(1 + \frac{s}{2}) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin(n\pi q)}{n\pi q} \times \frac{1}{n^s}} \qquad (6)$$

Corollary 1. Let $V \sim \mu_{a\sqrt{2},\pi\sqrt{2}}$ be independent of $Z \sim N(0,1)$. Then $X = Z\sqrt{V}$ is quasi logistic with parameter $\theta = \cos a$ and has a scale mixing Gaussian distribution.

Proof. If we take $V \sim \mu_{a,b}$ then for $t \in \mathbb{R}$ we have

$$\mathbb{E}(e^{itZ\sqrt{V}}) = \mathbb{E}(\mathbb{E}(e^{itZ\sqrt{V}}|V)) = \mathbb{E}(e^{-t^2V/2}) = \frac{b\sinh(at/\sqrt{2})}{a\sinh(bt/\sqrt{2})}$$

In particular replacing (a,b) by $(a\sqrt{2},\pi\sqrt{2})$ and using the first Theorem we get the result.

Corollary 2. Suppose that $V \sim \mu_{\pi q,\pi}$ and $Y = \sqrt{V}$ with a QKS distribution. Then

$$\Pr(Y > y) = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(\pi qn)}{\pi qn}e^{-n^2y^2}.$$

$q = 0$ gives the classical Kolmogorov Smirnov distribution.

Proof of the existence of $\mu_{a,b}$

Proof of Theorem 3.1. We use

$$\frac{b \sinh a\sqrt{s}}{a \sinh b\sqrt{s}} = \prod_{n=1}^{\infty} \left(\frac{1 + \frac{a^2 s}{\pi^2 n^2}}{1 + \frac{b^2 s}{\pi^2 n^2}} \right). \tag{7}$$

With the definition of (ϵ_n, W_n) we write

$$\frac{1 + \frac{a^2 s}{\pi^2 n^2}}{1 + \frac{b^2 s}{\pi^2 n^2}} = q^2 + \left(1 - q^2\right) \frac{1}{1 + \frac{b^2 s}{\pi^2 n^2}} = \mathbb{E}(e^{-s \frac{b^2}{\pi^2 n^2} \epsilon_n W_n}) \tag{8}$$

From the convergence theorem of Laplace transforms the existence of $\mu_{a,b}$ is proved.

Calculation of the density of $\mu_{a,b}$, first proof

We first give the Mellin transform of V and we will get the density of V from its Mellin transform. We have seen in part 1) that $V \sim \mu_{\pi q,\pi}$ and that

$$\mathbb{E}(e^{-sV}) = \frac{1}{q}\frac{\sinh \pi q\sqrt{s}}{\sinh \pi\sqrt{s}}.$$

We now use part 2) of Theorem 2.1, by considering X_θ with $\theta = \cos \pi q$, and the Gaussian random variable $Z \sim N(0,1)$ independent of V:

$$\mathbb{E}(e^{itZ\sqrt{2V}}) = \mathbb{E}(\mathbb{E}(e^{itZ\sqrt{2V}})|V) = \mathbb{E}(e^{-t^2V}) = \frac{1}{q}\frac{\sinh \pi qt}{\sinh \pi t} = \mathbb{E}(e^{itX_\theta})$$

which implies $X_\theta = Z\sqrt{2V}$.

Calculation of the density of $\mu_{a,b}$, continuation of the first proof

Recall that Z^2 is χ^2_1 distributed: this implies that

$$\mathbb{E}(Z^{2s}) = 2^s \frac{\Gamma(s+\frac{1}{2})}{\sqrt{\pi}}, \quad \mathbb{E}(|Z|^s) = 2^{s/2} \frac{\Gamma(\frac{1+s}{2})}{\sqrt{\pi}}.$$

Recall also the duplication formula

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$$

that we are going to apply to $z = (1+s)/2$. For convenience we write

$$K(s) = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sinh \pi nq}{\pi nq}\frac{1}{n^s}.$$

From the Mellin transform obtained in Theorem 1 we have $\mathbb{E}(|X|^s) = \Gamma(1+s)K(s)$.

Calculation of the density of $\mu_{a,b}$, end of the first proof

Since $|X| = |Z|\sqrt{2V}$ we obtain

$$\begin{aligned}\mathbb{E}((\sqrt{V})^s) &= \frac{\mathbb{E}(|X|^s)}{2^{s/2}\mathbb{E}(|Z|^s)} \\ &= \Gamma(1+s)K(s) \times 2^{-s/2}\frac{\sqrt{\pi}}{2^{s/2}\Gamma(\frac{1+s}{2})} = \Gamma(1+\frac{s}{2})K(s),\end{aligned}$$

this proves (6). From this we can write

$$\begin{aligned}\mathbb{E}(V^s) &= 2\Gamma(1+s)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sinh \pi nq}{\pi nq}\frac{1}{n^{2s}} \\ &= 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sinh \pi nq}{\pi nq}\int_0^{\infty} n^2 e^{-n^2} v^s dv.\end{aligned}$$

We have proved that $\mathbb{E}(V^s) = \int_0^{\infty} v^s g(v)dv$ which implies that g is the density of V.

Calculation of the density of $\mu_{a,b}$, second proof

STEP 1: Decomposition in partial fractions of a rational fraction: if $c_1, \ldots, c_N, \ldots$ are positive distinct numbers then

$$\frac{1}{\prod_{n=1}^{N}(1+c_n s)} = \sum_{n=1}^{N} \frac{1}{\prod_{j\neq n, 1\leq j\leq N}(1-\frac{c_j}{c_n})} \times \frac{1}{1+c_n s} \tag{9}$$

Approximation $g^{(N)}$ of the density g of $\mu_{a,b}$

STEP 2: We now compute an approximation of the density g of V. To do this we introduce the partial sums

$$V_N = \sum_{n=1}^{N} \frac{\epsilon_n W_n}{n^2}$$

the density $g^{(N)}(v)$ of V_N and the density $g_\epsilon^{(N)}(v)$ of V_N conditioned by $\epsilon = (\epsilon_n)_{n\geq 1}$. We now apply (9) to the particular case $c_n = \epsilon_n/n^2$ and we obtain

$$g_\epsilon^{(N)}(v) = \sum_{n=1}^{N} \frac{\epsilon_n}{\prod_{j\neq n, 1\leq j\leq N}(1 - \frac{\epsilon_j n^2}{j^2})} \times n^2 e^{-n^2 v} \qquad (10)$$

Since $\epsilon = (\epsilon_n)_{1\leq n\leq N}$ takes only a finite number of values we can claim that $g^{(N)}(v) = \mathbb{E}(g_\epsilon^{(N)}(v))$.

Approximation : continuation

$$\mathbb{E}\left(\frac{1}{1-\epsilon_j\frac{n^2}{j^2}}\right)=\frac{1-\frac{a^2n^2}{b^2j^2}}{1-\frac{n^2}{j^2}}.$$

With the following notation

$$\boxed{u_q^{(N)}(n)=(1-q^2)\prod_{j\neq n,1\leq j\leq N}\frac{1-\frac{q^2n^2}{j^2}}{1-\frac{n^2}{j^2}}}$$

and using the independence of the ϵ_j's we have

$$g^{(N)}(v)=\sum_{n=1}^{N}u_q^{(N)}(n)\times n^2e^{-n^2v}.$$

An elegant limit

STEP 3: We compute $\lim_{N\to\infty} u_q^{(N)}(n)$. Numerator:

$$\lim_{N\to\infty} \prod_{j\neq n, 1\leq j\leq N} (1-\frac{q^2n^2}{j^2}) = \frac{1}{\pi qn} \times \sin(\pi qn).$$

Denominator: to compute $\lim_{N\to\infty} \prod_{j\neq n,1\leq j\leq N}(1-\frac{n^2}{j^2})$ we use the following elementary calculation

$$\prod_{j\neq n}(1-\frac{z^2}{j^2}) = \frac{\sin \pi z}{\pi z} \times \frac{1}{1-\frac{z^2}{n^2}} \to_{z\to n} \frac{(-1)^{n-1}}{2}.$$

leading to $\boxed{\lim_{N\to\infty} u_q^{(N)}(n) = (-1)^{n-1}\frac{2}{\pi qn}\sin(\pi qn)}$.

With uniform convergence we arrive at

$$g(v) = \frac{2}{\pi q}\sum_{n=1}^{\infty}(-1)^{n+1}\sin(\pi nq) \times ne^{-n^2v} \qquad (11)$$

Another topic on deconvolution in several dimensions.

In one dimension we have seen that $X = \sqrt{V}Z$ implies that the law of $V > 0$ is known if the law of X is known. This is not true anymore for dimension ≥ 2.

Theorem 3. Let A be a random nonsingular square matrix of order n, independent of $Z \in \mathbb{R}^n \setminus \{0\}$ and such that $uZ \sim Z$ for all $u \in \mathbb{O}(n)$. Let $V = AA^*$. Then the following holds.

1. $AZ \sim V^{1/2}Z$, that is, if we replace $V^{1/2}$ by any generalized square root A of V, the distribution of AZ remains the same.
2. If $AZ \sim Z$ then $\Pr(V = I_n) = 1$. In other terms, $AZ \sim Z$ if and only if $\Pr(AA^* = I_n) = 1$, i.e $A \in \mathbb{O}(n)$ almost surely.

Proof. Let us skip the proof of part 1), no new ideas for it.

$AZ \sim Z \Leftrightarrow A$ is almost surely orthogonal

To prove 2., consider also $\varphi(s) = \mathbb{E}(e^{i\langle s,Z\rangle})$. Since $uZ \sim Z$ for all $u \in \mathbb{O}(n)$ there exists a real function g defined on $[0,\infty)$ such that $\varphi(s) = g(\|s\|^2)$. Since $Z \sim AZ$ we can write

$$g(\|s\|^2) = \mathbb{E}(g(s^* Vs)) \ . \tag{12}$$

Next, observe that if $R \geq 0$ is independent of $Z = (Z_1, \ldots, Z_n)$ and if $Z_1 R \sim Z_1$ then $\Pr(R = 1) = 1$: just check the characteristic functions of the log.

Continuation of $AZ \sim Z \Leftrightarrow A$ orthogonal

Now denote $V = (V_{ij})_{1 \leq i,j \leq n}$ and apply the above observation to $R = \sqrt{V_{11}}$ by taking $s = (t, 0, \dots, 0)$ in (12). We obtain

$$\mathbb{E}(e^{itZ_1}) = \varphi((t, 0, \dots, 0)) = g(t^2) = \mathbb{E}(g(t^2 V_{11})) = \mathbb{E}(e^{it\sqrt{V_{11}}Z_1})$$

which implies $Z_1 \sim V_{11} Z_1$ and $\Pr(V_{11} = 1) = 1$. Similarly $\Pr(V_{ii} = 1) = 1$ for all $i = 2, \cdots, n$.

End of $AZ \sim Z \Leftrightarrow A$ orthogonal

Finally, we consider $R = \sqrt{1 + V_{12}}$ and we take $s = (t/\sqrt{2}, t/\sqrt{2}, \ldots, 0)$ in (12). Using the fact that $(Z_1 + Z_2)/\sqrt{2} \sim Z_1$ we write

$$\begin{aligned} \mathbb{E}(e^{itZ_1}) &= \mathbb{E}(e^{it(Z_1+Z_2)/\sqrt{2}}) = \varphi((t/\sqrt{2}, t/\sqrt{2}, \ldots, 0)) \\ &= \mathbb{E}(g(\frac{1}{2}\,t^2(V_{11} + V_{22} + 2V_{12})) = \mathbb{E}(g(t^2(1 + V_{12})) \\ &= \mathbb{E}(e^{itZ_1\sqrt{1+V_{12}}}) \end{aligned}$$

and we get $\Pr(V_{12} = 0) = 1$. Similarly $\Pr(V_{ij} = 0) = 1$ for $i \neq j$ and finally $\Pr(V = I_n) = 1$ as desired. $\square$

Non identifiability in dimension ≥ 2

It is not difficult to choose a gamma distribution for the scalar V_1^{-1} and a Wishart distribution for the positive definite matrix V^{-1} to get that

$$\sqrt{V}Z \sim \sqrt{V_1}Z$$

Approximation of the density of X by a Gaussian density

In some practical applications, the distribution of V is not very well known, and it is interesting to replace the density f of $X = \sqrt{V}Z$ by the density of an ordinary normal distribution $N(0, t_0)$. The $L^2(\mathbb{R})$ distance is well adapted to this problem.

A list of facts about the L^2 approximation

Theorem 4.
1) $f \in L^2(\mathbb{R})$ if and only if

$$\mathbb{E}\left(\frac{1}{\sqrt{V+V_1}}\right) < \infty$$

when V and V_1 are independent with the same distribution μ.
2) If $f \in L^2(\mathbb{R})$, there exists a unique $t_0 = t_0(\mu) > 0$ which minimizes

$$t \mapsto I_V(t) = \int_{-\infty}^{\infty} \left[f(x) - \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \right]^2 dx.$$

Continuation of facts about the L^2 approximation

3)The scalar $y_0 = 1/t_0$ the unique positive solution of the equation

$$\int_0^\infty \frac{\mu(dv)}{(1+vy)^{3/2}} = \frac{1}{2^{3/2}}. \qquad (13)$$

4) The value of $I_V(t_0)$ is

$$I_V(t_0) = \sqrt{\frac{2}{\pi}} \left(\mathbb{E}\left(\frac{1}{\sqrt{V+V_1}} \right) - 2\mathbb{E}\left(\frac{1}{\sqrt{V+t_0}} \right) + \frac{1}{\sqrt{2t_0}} \right)$$

5) Finally $t_0 \leq \mathbb{E}(V)$.

Reference

Details about L^2 theory and about some of the above topics can be found in

Gaussian approximation of Gaussian scale mixtures,
Gérard Letac and Hélène Massam
Kybernetika 2020, ArXiv 1810.02036

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