LARGE DEVIATIONS FOR THE WIENER SAUSAGE

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§ OUTLINE

In this mini-course we consider the Wiener sausage:

$$W_t = \bigcup_{s\in[0,t]} B_1(\beta_s), \qquad t \geq 0.$$

Here, $(\beta_s)_{s\geq 0}$ is Brownian motion on $\mathbb{R}^d$, and $B_1(x)$ is the closed ball of radius 1 centred at $x \in \mathbb{R}^d$. Our focus will be on the volume and the capacity of W_t as $t \to \infty$.

Lecture 1:
Basic facts, LLN, CLT

Lecture 2:
LDP

Lectures 3 + 4:
Sketch of the proof of the LDP: skeleton approach

LECTURE 1

Basic facts, LLN, CLT

§ WIENER SAUSAGE

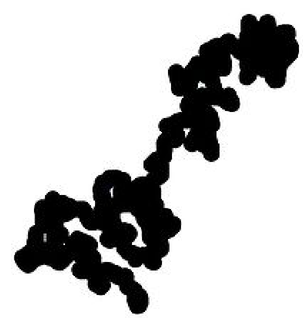
Let $\beta = (\beta_s)_{s \geq 0}$ be Brownian motion on $\mathbb{R}^d$, i.e., the continuous-time Markov process with generator Δ (the Laplace operator). Write $\mathbb{P}_x$ to denote the law of β starting from x, and put $\mathbb{P} = \mathbb{P}_0$.

The Wiener sausage at time t is the random set

$$W_t = \bigcup_{s \in [0,t]} B_1(\beta_s), \qquad t \geq 0,$$

i.e., the Brownian motion drags around a ball of radius 1, which traces out a sausage-like environment.

Mark Kac 1974



The Wiener sausage is an important object because it is one of the simplest examples of a non-Markovian functional of Brownian motion. It plays a key role in the study of various stochastic phenomena:

- heat conduction
- trapping in random media
- spectral properties of random Schrödinger operators
- Bose-Einstein condensation

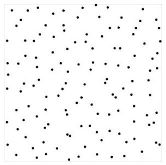
We will look at two specific quantities:

$$\mathcal{V}_t = \mathrm{Vol}(W_t),$$
$$\mathcal{C}_t = \mathrm{Cap}(W_t).$$

Here, Vol stands for volume and Cap for capacity. Only the case $d \geq 2$ is interesting.

▶ The volume plays a role in trapping phenomena. Write PPP_α to denote the law of a Poisson Point Process on $\mathbb{R}^d$ with intensity $\alpha \in (0,\infty)$. Place balls of radius 1 around the points. Let τ be the first time that β hits a ball. Then

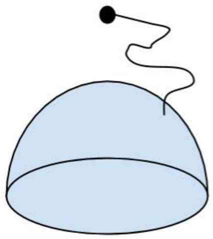
$$\mathsf{PPP}_\alpha(\tau > t) = \mathrm{e}^{-\alpha \mathcal{V}_t}, \qquad t \geq 0.$$



▶ The capacity plays a role in hitting phenomena. Let $\bar\tau_{W_t}$ be the first time an auxiliary (!) Brownian motion $\bar\beta$ hits W_t. Then

$$\lim_{|x|\to\infty} |x|^{d-2}\, \bar{\mathbb{P}}_x\Big(\bar\tau_{W_t} < \infty\Big) = \frac{\mathcal{C}_t}{\kappa_d}, \qquad t \geq 0, \quad d \geq 3,$$

with $\kappa_d = \mathrm{Cap}(B_1(0))$.



Another interpretation is that $\kappa_d/\mathcal{C}_t$ equals the minimal electrostatic energy of a unit charge distributed on W_t.

Other set functions are of interest too:

perimeter, moment of inertia, principal Dirichlet eigenvalue, heat content, torsional rigidity.

Many have the property that they are either minimal or maximal when the set is a ball, which is analytically helpful.

In what follows we recall some basic facts, and state LLNs and CLTs. After that we focus on LDPs. It will turn out that there is a remarkable dependence on the dimension d.

NOTE: By Brownian scaling, it is trivial to transfer all the properties to be described below to the Wiener sausage with radius $r \in (0, \infty)$ rather than $r = 1$.

§ VOLUME

The following strong LLN holds:

$$d \geq 3: \quad \lim_{t\to\infty} \frac{1}{t}\mathcal{V}_t = \kappa_d \quad \mathbb{P}-a.s.$$

Spitzer 1964

where

$$\kappa_d = \mathrm{Cap}(B_1(0)) = \frac{4\pi^{d/2}}{\Gamma(\frac{d-2}{2})}.$$

The existence of the limit follows from the subaddivity property

$$\mathrm{Vol}(A \cup B) \leq \mathrm{Vol}(A) + \mathrm{Vol}(B) \qquad \forall A, B \subset \mathbb{R}^d \text{ Borel.}$$

The identification of the limit requires potential theory.

The case $d = 2$ is critical:

$$\lim_{t\to\infty} \frac{\log t}{t} \mathcal{V}_t = \kappa_2 = 4\pi \quad \mathbb{P} - a.s.$$

Le Gall 1988

The CLT holds with

$$\mathbb{V}\mathrm{ar}(\mathcal{V}_t) \asymp \begin{cases} t^2/\log^4 t, & d = 2, \\ t \log t, & d = 3, \\ t, & d \geq 4, \end{cases}$$

Spitzer 1964, Le Gall 1988

The limit law is Gaussian for $d \geq 3$ and non-Gaussian for $d = 2$.

§ CAPACITY

The (Newtonian) capacity of a Borel set $A \subset \mathbb{R}^d$ can be defined through the variational formula

$$\boxed{\frac{1}{\mathsf{Cap}(A)} = \inf_{\mu \in \mathcal{P}(A)} \int_A \mu(\mathrm{d}x) \int_A \mu(\mathrm{d}y)\, G_d(x,y),}$$

where $\mathcal{P}(A)$ is the set of probability measures on A, and

$$G_d(x,y) = \frac{1}{\kappa_d |x-y|^{d-2}}, \qquad x, y \in \mathbb{R}^d,$$

is the Green function. Think of the above integral as the electrostatic energy when the set A is a conductor with a unit charge.

A different formula is needed for the case $d = 2$, which we will not consider.

The following strong LLN holds:

$$d \geq 5: \quad \lim_{t\to\infty} \frac{1}{t}\mathcal{C}_t = c_d \quad \mathbb{P}-a.s.$$

Asselah, Schapira, Sousi 2018

The existence of the limit follows from the subaddivity property

$$\text{Cap}(A \cup B) \leq \text{Cap}(A) + \text{Cap}(B) \qquad \forall A, B \subset \mathbb{R}^d \text{ Borel.}$$

The fact that the limit is non-degenerate requires some estimates. No explicit formula is known for c_d.

The case $d = 4$ is critical:

$$\lim_{t\to\infty} \frac{\log t}{t} \mathcal{C}_t = c_4 = 8\pi^2 \quad \mathbb{P} - a.s.$$

Asselah, Schapira, Sousi 2019

The CLT is expected (!) to hold with

$$\mathbb{V}\mathrm{ar}(\mathcal{C}_t) \asymp \begin{cases} t^2/\log^4 t, & d = 4, \\ t \log t, & d = 5, \\ t, & d \geq 6, \end{cases}$$

Asselah, Schapira, Sousi 2018 – discrete setting

The limit law is expected (!) to be Gaussian for $d \geq 5$ and non-Gaussian for $d = 4$.

The cases $d = 2, 3$ are interesting too, but are less well understood. We will not consider them.

LECTURE 2

LDP

§ LARGE DEVIATIONS FOR THE VOLUME

THEOREM 1 Van den Berg, Bolthausen, dH, 2001 + 2004

Let $d \geq 3$. *For every* $b > 0$*,*

$$\lim_{t\to\infty} \frac{1}{t^{(d-2)/d}} \log \mathbb{P}(\mathcal{V}_t \leq bt) = -I_d^{\downarrow}(b),$$

where

$$I_d^{\downarrow}(b) = \inf_{\phi\in\Phi_d(b)} \int_{\mathbb{R}^d} |\nabla\phi|^2(x)\,\mathrm{d}x$$

with

$$\Phi_d(b) = \Big\{\phi \in H^1(\mathbb{R}^d)\colon\ \int_{\mathbb{R}^d} \phi^2(x)\,\mathrm{d}x = 1, \ \int_{\mathbb{R}^d} \Big(1 - \mathrm{e}^{-\kappa_d \phi^2(x)}\Big)\,\mathrm{d}x \leq b\Big\}.$$

HEURISTICS:

The idea is that the optimal strategy for β to realise the event $\{\mathcal{V}_t \leq bt\}$ is to behave like a Brownian motion in a drift field

$$t^{1/d}x \mapsto (\nabla\phi/\phi)(x) \qquad \text{for some } \phi \in \Phi_d(b).$$

The cost of adopting this drift during a time t is

$$\exp[t^{(d-2)/d}\,\|\nabla\phi\|_2^2],$$

to leading order. Conditional on adopting this drift, the path spends time $t\phi^2(x)\mathrm{d}x$ in the volume element $t^{1/d}\mathrm{d}x$ and the Wiener sausage covers a fraction

$$1 - \mathrm{e}^{-\kappa_d \phi^2(x)}$$

of this volume element.

CONCLUSION:

The optimal strategy for the Wiener Sausage to achieve a downward large deviation of its volume is to look like a Swiss cheese!

THEOREM 2 Van den Berg, Bolthausen, dH, 2001 + 2004

Let $d = 2$. *For every* $b > 0$,

$$\lim_{t\to\infty} \frac{1}{\log t} \log \mathbb{P}(\mathcal{V}_t \leq bt/\log t) = -I_2^{\downarrow}(b),$$

where

$$I_2^{\downarrow}(b) = \inf_{\phi\in\Phi_2(b)} \int_{\mathbb{R}^2} |\nabla\phi|^2(x)\,\mathrm{d}x$$

with

$$\Phi_2(b) = \Big\{\phi \in H^1(\mathbb{R}^2)\colon\ \int_{\mathbb{R}^2} \phi(x)^2\,\mathrm{d}x = 1, \\ \int_{\mathbb{R}^2} \Big(1 - \mathrm{e}^{-\kappa_2 \phi(x)^2}\Big)\,\mathrm{d}x \leq b\Big\}.$$

NOTE:

The cost of a downward large deviation in the critical dimension $d = 2$ is polynomial rather than stretched exponential.

§ RATE FUNCTIONS

By Brownian scaling, the variational formula for the rate function can be standardised, namely,

$$I_d^{\downarrow}(b) = \kappa_d^{-2/d}\,\chi_d\left(\frac{b}{\kappa_d}\right), \qquad b > 0,$$

where

$$\chi_d(u) = \inf\Big\{\|\nabla\psi\|_2^2\colon\ \psi \in H^1(\mathbb{R}^d),\ \|\psi\|_2 = 1,\ \int_{\mathbb{R}^d}(1-\mathrm{e}^{-\psi^2}) \leq u\Big\}.$$

The function χ_d is continuous on $(0,\infty)$, strictly decreasing on $(0,1)$, and equal to zero on $[1,\infty)$. If a minimiser exists, then it is unique modulo translations, radially symmetric and radially strictly decreasing.

Moreover,

$$\lim_{u \downarrow 0} u^{2/d} \chi_d(u) = -\lambda_d,$$

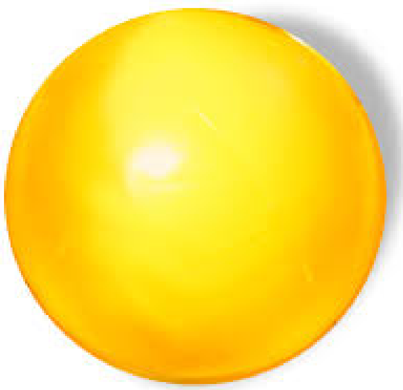
with $\lambda_d \in (0, \infty)$ the principal Dirichlet eigenvalue of $-\Delta$ on $B_1(0)$.

The latter corresponds to the regime where $\mathcal{V}_t = o(\mathbb{E}(\mathcal{V}_t))$ and the Swiss cheese squeezes out its holes to becomes a ball. This regime was studied earlier and is much easier to handle.

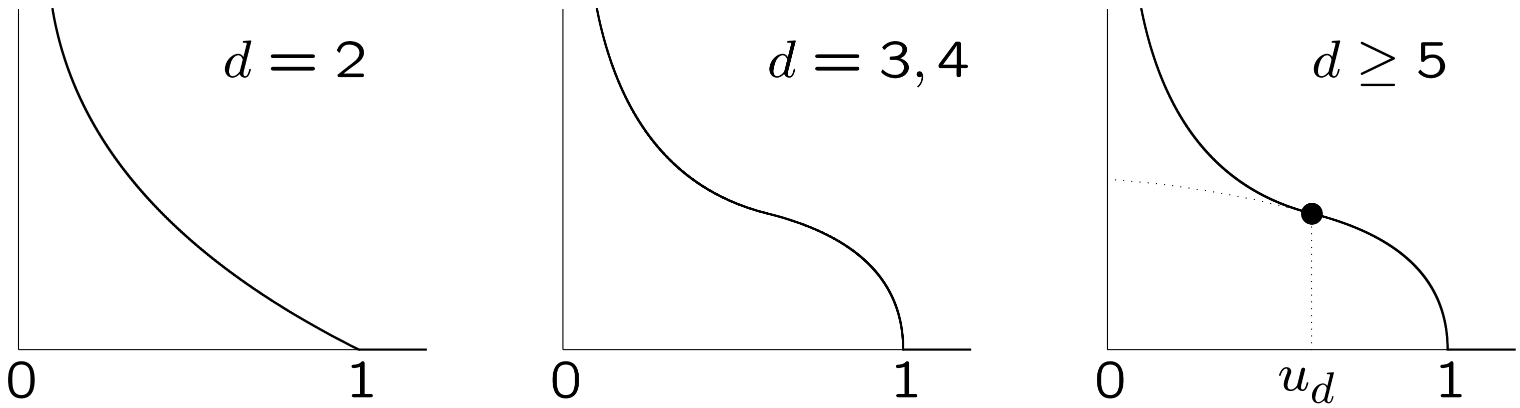
Donsker, Varadhan 1975

Bolthausen 1990 + 1994

Sznitman 1990



Qualitative picture of $u \mapsto \chi_d(u)$:



For $d \geq 5$, there is a critical value $u_d \in (0, 1)$ above which the variational formula has no minimiser: mass leaks a way to infinity.

The optimal strategy is time-inhomogeneous: partly on scale $t^{1/d}$, partly on scale $\sqrt{t}$.

§ WHAT ABOUT UPWARD LARGE DEVIATIONS?

Upward large deviations are more costly. For $d \geq 3$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\mathcal{V}_t \geq bt) = -I_d^{\uparrow}(b), \qquad b > 0,$$

Hamana, Kesten 2001

with

$$I_d^{\uparrow}(b) > 0 \qquad \forall\, b > \kappa_d.$$

Upward large deviations are much harder to capture than downward large deviations: they require a local dilation of the Wiener sausage essentially everywhere.

No variational formula is known for $I_d^{\uparrow}$. Only bounds are available via estimates on exponential moments.

van den Berg, Tóth 1991

van den Berg, Bolthausen 1994

§ LARGE DEVIATIONS FOR THE CAPACITY

THEOREM 3 Van den Berg, Bolthausen, dH, in progress

Let $d \geq 5$. *For every* $b > 0$,

$$\lim_{t\to\infty} \frac{1}{t^{(d-4)/(d-2)}} \log \mathbb{P}(\mathcal{C}_t \leq bt) = -I_d^{\downarrow}(b),$$

where

$$I_d^{\downarrow}(b) = \inf_{\phi \in \Phi_d(b)} \int_{\mathbb{R}^d} |\nabla \phi|^2(x)\,\mathrm{d}x$$

with

$$\Phi_d(b) = \Big\{ \phi \in H^1(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} \phi^2(x)\,\mathrm{d}x = 1, \int_{\mathbb{R}^d} \phi^2(x)\,[c_d u_\phi(x)]\,\mathrm{d}x \leq b \Big\},$$

LDP

where u_ϕ is the unique solution of the equation

$$\Delta u = \left(\frac{c_d}{2d}\phi^2\right) u, \qquad u(\infty) \equiv 1,$$

Note that the function u_ϕ is the solution of the Schrödinger equation with potential $-\frac{c_d}{2d}\phi^2$ and with boundary condition 1 at infinity.

REMARK: Rough bounds for the large deviations of the capacity of simple random walk on $\mathbb{Z}^d$ with $d \geq 5$ were derived in Asselah, Schapira 2020. These bounds capture the right order of magnitude, but are off in the constants, and are valid only for large deviations that are near the average capacity.

HEURISTICS:

The idea is that the optimal strategy for β to realise the event $\{\mathcal{C}_t \leq bt\}$ is to behave like a Brownian motion in a drift field

$$t^{1/(d-2)}x \mapsto (\nabla\phi/\phi)(x) \qquad \text{for some } \phi \in \Phi_d(b).$$

The cost of adopting this drift during a time t is

$$\exp[t^{(d-4)/(d-2)}\,\|\nabla\phi\|_2^2],$$

to leading order. Conditional on adopting this drift, the path spends time $t\phi^2(x)\mathrm{d}x$ in the volume element $t^{1/(d-2)}\mathrm{d}x$ and the capacity associated with the Wiener sausage in this volume element is

$$u_\phi(x)c_d\,[t\phi^2(x)\mathrm{d}x].$$

In the latter expression:

- c_d is the probability that $\bar{\beta}$ escapes locally from W_t, i.e., moves out of the volume element $t^{1/(d-2)}\mathrm{d}x$ without hitting W_t.

- $u_\phi(x)$ is the probability that β escapes globally from W_t, i.e., moves to infinity without hitting the part of W_t that lies outside the volume element $t^{1/(d-2)}\mathrm{d}x$.

For capacity both local and global properties control the downward large deviations.

CONCLUSION:

The optimal strategy for the Wiener Sausage to achieve a downward large deviation of its capacity is to look like thin Italian spaghetti!

THEOREM 4 Van den Berg, Bolthausen, dH, in progress

Let $d = 4$. *For every* $b > 0$,

$$\lim_{t\to\infty} \frac{1}{\log t} \log \mathbb{P}(\mathcal{C}_t \leq bt/\log t) = -I_4^{\downarrow}(b),$$

where

$$I_4^{\downarrow}(b) = \inf_{\phi\in\Phi_4(b)} \int_{\mathbb{R}^4} |\nabla\phi|^2(x)\, \mathrm{d}x$$

with

$$\Phi_4(b) = \Big\{\phi \in H^1(\mathbb{R}^4)\colon \int_{\mathbb{R}^4} \phi(x)^2\, \mathrm{d}x = 1, \\ \int_{\mathbb{R}^4} \phi(x)^2 \, [c_4 u_\phi(x)]\, \mathrm{d}x \leq b\Big\},$$

where u_ϕ *is the unique solution of the equation*

LDP

$$\Delta u = \left(\frac{c_4}{8}\phi^2\right) u, \qquad u(\infty) \equiv 1.$$

The cost of a downward large deviation in the critical dimension $d = 4$ is polynomial rather than stretched exponential.

§ RATE FUNCTIONS

By Brownian scaling, we get

$$I_d^{\downarrow}(b) = \left(\frac{c_d}{2d}\right)^{-2/(d-2)} \chi_d\left(\frac{b}{c_d}\right), \qquad b > 0,$$

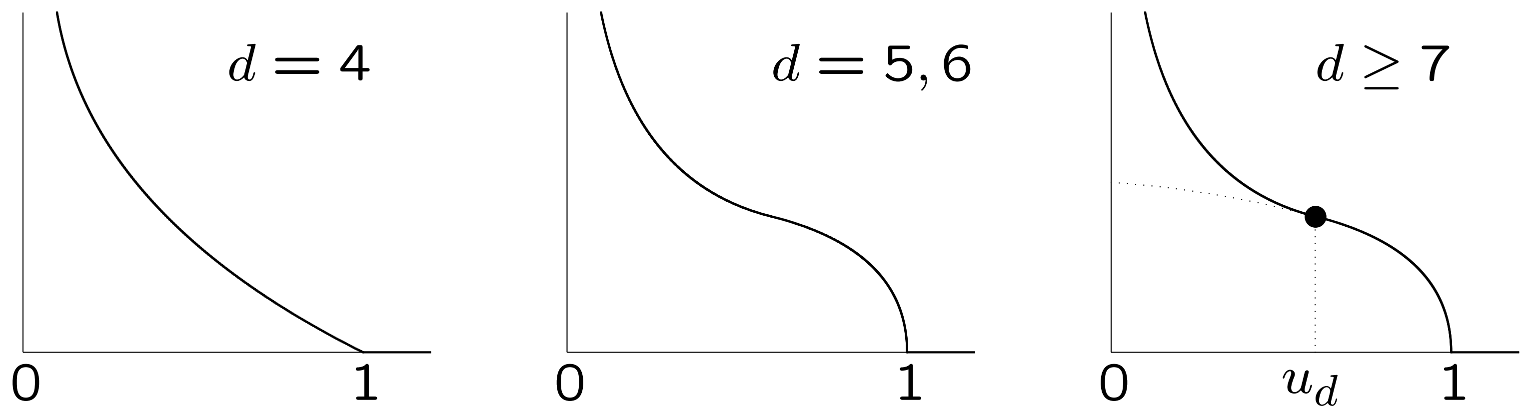
where

$$\chi_d(u) = \inf\Big\{\|\nabla\psi\|_2^2\colon\ \psi \in H^1(\mathbb{R}^d),\ \|\psi\|_2 = 1,\ \textstyle\int_{\mathbb{R}^d} \psi^2 u_\psi \le u\Big\}$$

with u_ψ the solution of $\Delta u = \psi^2 u$, $u(\infty) \equiv 1$.

We expect that upward large deviations are exponentially costly, but no results are yet available.

Qualitative picture of $u \mapsto \chi_d(u)$:



REMARKABLE: The scaled rate function for the capacity in dimension $d+2$ is qualitatively the same as that for the volume in dimension d.

LECTURE 3

Sketch of the proof of the LDP for the volume:
skeleton approach

§ UPPER BOUND

We prove the upper bound for the volume:

$$\limsup_{t\to\infty} \frac{1}{t^{(d-2)/d}} \log \mathbb{P}(\mathcal{V}_t \leq bt) \leq -I_d^{\downarrow}(b), \quad b > 0, \quad d \geq 3.$$

The lower bound follows after an easy modification, as we explain later. Afterwards we consider the capacity.

The proof comes in 6 steps:

(1) Compactification
(2) Excursion decomposition
(3) Concentration
(4) Donsker-Varadhan LDP
(5) Short excursion limit
(6) Decompactification

(1) COMPACTIFICATION

Pick $N > 0$, and let $\Lambda_N = [-\frac{N}{2}, \frac{N}{2}]^d$. Clearly,

$$\mathrm{Vol}(W_{N,t}) \leq \mathrm{Vol}(W_t)$$

with $W_{N,t}$ the Wiener sausage of the Brownian motion wrapped around $t^{1/d}\Lambda_N$.

By Brownian scaling,

$$t^{-1}\mathrm{Vol}(W_{N,t}) =^{\mathrm{law}} \mathrm{Vol}(W^{\tau}_{N,\tau})$$

with

$$W^{\tau}_{N,\tau} = \cup_{s\in[0,\tau]} B_{r_\tau}(\beta_{N,s}),$$

where

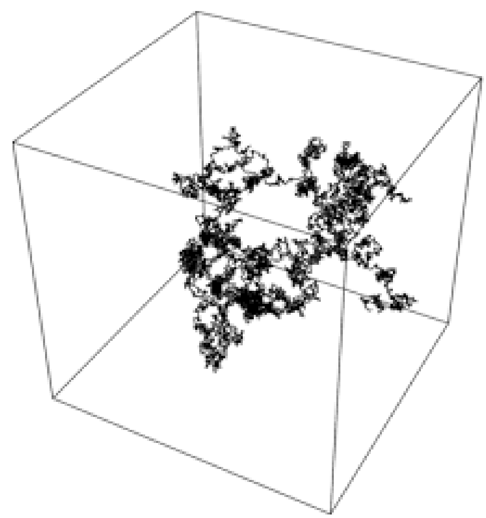
$$\tau = t^{(d-2)/d}, \quad r_\tau = \tau^{-1/(d-2)},$$

and β_N is Brownian motion wrapped around Λ_N. Hence, putting $\mathcal{V}_{N,\tau} = \mathrm{Vol}(W^{\tau}_{N,\tau})$, we have

$$\mathbb{P}(\mathcal{V}_t \le bt) \le \mathbb{P}_N(\mathcal{V}_{N,\tau} \le b)$$

where $\mathbb{P}_N$ is the law of β_N starting from 0.

The right-hand side involves the Wiener sausage on Λ_N with a radius r_τ that shrinks with τ.



The desired upper bound follows from the following two propositions.

PROPOSITION 1

Let $d \geq 3$. *For every* $b > 0$,

$$\lim_{\tau\to\infty} \frac{1}{\tau} \log \mathbb{P}_N(\mathcal{V}_{N,\tau} \leq b) = -I^{\downarrow}_{d,N}(b)$$

with $I^{\downarrow}_{d,N}(b)$ *given by the same variational formula as* $I^{\downarrow}_{d}(b)$, *except that* $\mathbb{R}^d$ *is replaced by* Λ_N *with a periodic boundary.*

PROPOSITION 2

$\lim_{N\to\infty} I^{\downarrow}_{d,N}(b) = I^{\downarrow}_{d}(b)$ *for every* $b > 0$.

(2) EXCURSION DECOMPOSITION

To prove Proposition 1, we fix $N > 0$ and suppress it from the notation. Decompose β into excursions of length $\epsilon > 0$. Let

$$\boxed{\mathbb{X}_{\tau,\epsilon} = \{\beta_{i\epsilon}\}_{i=1}^{\tau/\epsilon}}$$

Excursions

denote the endpoints of the excursions, which we refer to as the skeleton. In what follows we will write

$$\mathbb{E}_{\tau,\epsilon}(\mathcal{V}_\tau) = \mathbb{E}(\mathcal{V}_\tau \mid \mathbb{X}_{\tau,\epsilon})$$

to denote the average of $\mathcal{V}_\tau$ conditional on $\mathbb{X}_{\tau,\epsilon}$.

Let

$$L_{\tau,\epsilon} = \frac{1}{\tau/\epsilon} \sum_{i=1}^{\tau/\epsilon} \delta_{(\beta_{(i-1)\epsilon}, \beta_{i\epsilon})}$$

denote the empirical pair distribution of the skeleton. Let W_i denote the Wiener sausage associated with the i-th excursion. We have

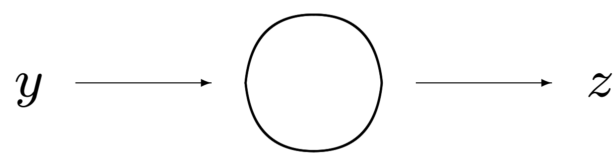
$$\begin{aligned}
\mathbb{E}_{\tau,\epsilon}(\mathcal{V}_\tau) &= \int_{\Lambda_N} \mathrm{d}x \; \mathbb{P}_{\tau,\epsilon}\Big(x \in \bigcup_{i=1}^{\tau/\epsilon} W_i \Big) \\
&=^{!} \int_{\Lambda_N} \mathrm{d}x \left(1 - \prod_{i=1}^{\tau/\epsilon} \left[1 - \mathbb{P}_{\tau,\epsilon}(x \in W_i) \right] \right) \\
&= \int_{\Lambda_N} \mathrm{d}x \left(1 - \exp\left[\frac{\tau}{\epsilon} \int_{\Lambda_N \times \Lambda_N} \right.\right. \\
&\qquad\qquad \left.\left. \log[1 - q_{\tau,\epsilon}(y-x, z-x)] \, L_{\tau,\epsilon}(\mathrm{d}y, \mathrm{d}z) \right] \right),
\end{aligned} \tag{$*$}$$

where

$$q_{\mathcal{T},\epsilon}(y,z) = \mathbb{P}_{y,z}\Big(\tau_{B_{r_{\mathcal{T}}}(0)} \leq \epsilon\Big)$$

with $\mathbb{P}_{y,z}$ the law of the Brownian Bridge of length ϵ starting at y and ending at z.

Through $(*)$, we have written $\mathbb{E}_{\mathcal{T},\epsilon}(\mathcal{V}_{\mathcal{T}})$ as an exponential functional of the random variable $L_{\mathcal{T},\epsilon}$. This functional also involves the Brownian Bridge function $q_{\mathcal{T},\epsilon}$, which for large $\mathcal{T}$ is small.



$B_{r_{\mathcal{T}}}(0)$

$0 < r_{\mathcal{T}} \ll 1$

(3) CONCENTRATION

LEMMA 1

For every $\delta > 0$,

$$\lim_{\epsilon \downarrow 0} \limsup_{\tau \to \infty} \frac{1}{\tau} \log \mathbb{P}\Big(|\mathcal{V}_\tau - \mathbb{E}_{\tau,\epsilon}(\mathcal{V}_\tau)| \geq \delta\Big) = -\infty.$$

PROOF: The proof uses concentration of measure. The key observation is that $\mathcal{V}_\tau = \mathrm{Vol}(\cup_{i=1}^{\tau/\epsilon} W_i)$ is a Lipschitz function of the W_i's, which are independent under the law $\mathbb{P}_{\tau,\epsilon}$.

Talagrand 1995 ■

(4) DONSKER-VARADHAN LDP

By the DV-LDP for the empirical pair distribution of a uniformly ergodic Markov chain, $(L_{\tau,\epsilon})_{\tau>0}$ satisfies the LDP on $\mathcal{P}(\Lambda_N \times \Lambda_N)$ with rate τ and with rate function $I_\epsilon^{(2)}/\epsilon$ given by

$$I_\epsilon^{(2)}(\mu) = \begin{cases} h(\mu \mid \mu_1 \otimes \pi_\epsilon), & \text{if } \mu_1 = \mu_2, \\ \infty, & \text{otherwise.} \end{cases}$$

Here, $h(\cdot \mid \cdot)$ denotes relative entropy, μ_1 and μ_2 are the left- and right-marginal of μ, and $\pi_\epsilon(x, \mathrm{d}y) = p_\epsilon(y - x)\mathrm{d}y$ is the time-ϵ Brownian transition kernel on Λ_N.

Donsker, Varadhan 1975

We can apply this LDP to $(*)$ to obtain the following.

LEMMA 2

$(\mathbb{E}_{\tau,\epsilon}(\mathcal{V}_\tau))_{\tau>0}$ *satisfies the LDP on* $[0,\infty)$ *with rate* τ *and with rate function* J_ϵ *given by*

$$J_\epsilon(b) = \inf\left\{\frac{1}{\epsilon} I_\epsilon^{(2)}(\mu)\colon\ \mu \in \mathcal{P}(\Lambda_N \times \Lambda_N), \Phi_\epsilon(\mu) = b\right\},$$

where

$$\Phi_\epsilon(\mu) = \int_{\Lambda_N} \mathrm{d}x \left(1 - \exp\left[-\frac{\kappa_d}{\epsilon}\int_{\Lambda_N\times\Lambda_N} \varphi_\epsilon(y-x, z-x)\, \mu(\mathrm{d}y, \mathrm{d}z)\right]\right)$$

with

$$\varphi_\epsilon(y,z) = \frac{\int_0^\epsilon ds\, p_s(-y)p_{\epsilon-s}(z)}{p_\epsilon(z-y)}.$$

PROOF: The key observation is the following fact:

$$\lim_{\tau \to \infty} |\tau q_{\tau,\epsilon}(y,z) - \kappa_d \varphi_\epsilon(y,z)| = 0$$

'uniformly' in $y, z \in \Lambda_N$.

Inserting this into $(*)$, and using that

$$\lim_{\tau \to \infty} q_{\tau,\epsilon}(y,z) = 0$$

'uniformly' in $y, z \in \Lambda_N$

to linearise the logarithm, we deduce the claim with the help of the contraction principle.

Some technicalities arise in order to deal with y, z that are either close to or far away from the origin on scale r_τ. ■

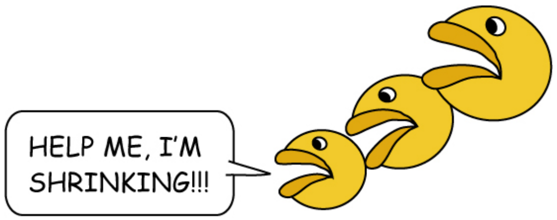
(5) SHORT EXCURSION LIMIT

Let

$$I_\epsilon^{(1)}(\nu) = \inf\{I_\epsilon^{(2)}(\mu)\colon\ \mu_1 = \nu\}$$

be the projection of $I_\epsilon^{(2)}$ onto $\mathcal{P}(\Lambda_N)$.

LEMMA 3



For every $\nu \in \mathcal{P}(\Lambda_N)$,

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} I_\epsilon^{(1)}(\nu) = I(\nu),$$

where

$$I(\nu) = \begin{cases} \int_{\Lambda_N} |\nabla \phi|^2(x)\, \mathrm{d}x, & \text{if } \frac{\mathrm{d}\nu}{\mathrm{d}x} = \phi^2 \text{ with } \phi \in H^1(\Lambda_N), \\ \infty, & \text{otherwise.} \end{cases}$$

PROOF: The representation formula

$$I_\epsilon^{(1)}(\nu) = -\inf_{0<u<\infty} \int_{\Lambda_N} \nu(\mathrm{d}x) \log\left(\frac{(\pi_\epsilon u)(x)}{u(x)}\right)$$

shows that $\epsilon \mapsto I_\epsilon^{(1)}(\nu)$ is subadditive for every ν. Hence the limit exists.

The limit can be identified by looking at the behaviour of π_ϵ, the time-τ Brownian transition kernel on Λ_N, as $\epsilon \downarrow 0$.

Donsker, Varadhan 1975 ■

I is the large deviation rate function for the empirical distribution of Brownian motion on Λ_N.

For $\nu \in \mathcal{P}(\Lambda_N)$, define

$$\Psi_\epsilon(\nu) = \int_{\Lambda_N} \mathrm{d}x \left(1 - \exp\left[-\frac{\kappa_d}{\epsilon}\int_0^\epsilon \mathrm{d}s \int_{\Lambda_N} p_s(x-y)\nu(\mathrm{d}y)\right]\right).$$

LEMMA 4

$$\lim_{\epsilon\downarrow 0} \sup_{\mu\in\mathcal{P}(\Lambda_N\times\Lambda_N):\ \frac{1}{\epsilon}I_\epsilon^{(2)}(\mu)<!\infty} |\Phi_\epsilon(\mu) - \Psi_\epsilon(\mu_1)| = 0.$$

PROOF: Estimate for $\mu \in \mathcal{P}(\Lambda_N \times \Lambda_N)$,

$$\begin{aligned} |\Phi_\epsilon(\mu) - \Psi_\epsilon(\mu_1)| &= |\Phi_\epsilon(\mu) - \Phi_\epsilon(\mu_1 \otimes \pi_\epsilon)| \\ &\leq \kappa_a \|\mu - \mu_1 \otimes \pi_\epsilon\|_{\mathrm{TV}} \end{aligned}$$

with TV the total variation norm. The claim follows from Pinsker's inequality:

$$\|\mu - \mu_1 \otimes \pi_\epsilon\|_{\mathrm{TV}} \leq 8\sqrt{I_\epsilon^{(2)}(\mu)}.$$

■

LEMMA 5

Define

$$\Gamma(f) = \int_{\Lambda_N} \mathrm{d}x \left(1 - \mathrm{e}^{-\kappa_d f(x)}\right), \qquad f \in L_1(\Lambda_N).$$

Then

$$\lim_{\epsilon \downarrow 0} \sup_{\nu \in \mathcal{P}(\Lambda_N):\ \frac{1}{\epsilon} I_\epsilon^{(1)}(\nu) <^{!} \infty} \left| \Gamma\left(\frac{\mathrm{d}\nu}{\mathrm{d}x}\right) - \Psi_\epsilon(\nu) \right| = 0,$$

PROOF: Estimate for $\nu \in \mathcal{P}(\Lambda_N)$,

$$\left| \Gamma\left(\frac{\mathrm{d}\nu}{\mathrm{d}x}\right) - \Psi_\epsilon(\nu) \right| \leq \int_{\Lambda_N} \mathrm{d}x \, \frac{\kappa_d}{\epsilon} \int_0^\epsilon \mathrm{d}s \left| \frac{d(\nu \otimes \pi_s)}{\mathrm{d}x} - \frac{\mathrm{d}\nu}{\mathrm{d}x} \right| (x)$$
$$= \frac{\kappa_d}{\epsilon} \int_0^\epsilon \mathrm{d}s \, \|\nu \otimes \pi_s - \nu\|_{\mathrm{TV}}.$$

The claim again follows from Pinsker's inequality. ■

We are now ready to prove Proposition 1.

PROOF:

For any test function $f\colon [0,\infty) \to \mathbb{R}$ that is bounded and continuous, we have, by Lemmas 1-5,

$$\lim_{\tau\to\infty} \frac{1}{\tau} \log \mathbb{E}\Big(\exp\left[\tau f(V_\tau)\right] \Big)$$

$$=^{L1} \lim_{\epsilon\downarrow 0} \lim_{\tau\to\infty} \frac{1}{\tau} \log \mathbb{E}\Big(\exp\Big[\tau f\Big(\mathbb{E}_{\tau,\epsilon}(V_\tau)\Big)\Big]\Big)$$

$$=^{L2} \lim_{\epsilon\downarrow 0} \sup_{\mu\in\mathcal{P}(\Lambda_N\times\Lambda_N)} \left\{ f(\Phi_\epsilon(\mu)) - \frac{1}{\epsilon} I_\epsilon^{(2)}(\mu) \right\}$$

$$=^{L4} \lim_{\epsilon\downarrow 0} \sup_{\mu\in\mathcal{P}(\Lambda_N\times\Lambda_N)} \left\{ f(\Psi_\epsilon(\mu_1)) - \frac{1}{\epsilon} I_\epsilon^{(2)}(\mu) \right\}$$

$$=^{L5} \lim_{\epsilon\downarrow 0} \sup_{\nu\in\mathcal{P}(\Lambda_N)} \left\{ f(\Psi_\epsilon(\nu)) - \frac{1}{\epsilon} I_\epsilon^{(1)}(\nu) \right\}$$

$$=^{L3} \sup_{\nu \in \mathcal{P}(\Lambda_N)} \left\{ f\left(\Gamma\left(\frac{\mathrm{d}\nu}{\mathrm{d}x}\right)\right) - I(\nu) \right\}$$

$$=^{L3} \sup_{\phi \in H^1(\Lambda_N):\ \|\phi\|_2^2 = 1} \left\{ f(\Gamma(\phi^2)) - \|\nabla\phi\|_2^2 \right\}.$$

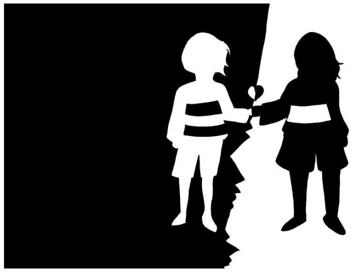
Since f is arbitrary, and

$$\Gamma(\phi^2) = \int_{\Lambda_N} \mathrm{d}x \left(1 - \mathrm{e}^{-\kappa_d \phi(x)^2}\right)$$

is the key integral for the constraint in our variational formula, the claim in Proposition 1 follows from Bryc's Lemma:

Moment-Generating Function $\Longleftrightarrow$ Rate Function

Bryc 1990



■

(6) DECOMPACTIFICATION

It remains to prove Proposition 2.

PROOF: With the help of convexity properties of the three integrals appearing in the variational formula for the rate function, it is not hard to prove that, as $\Lambda_N \to \mathbb{R}^d$, the one on Λ_N converges to the one on $\mathbb{R}^d$. ■

§ LOWER BOUND

PROOF: Let $C_{N,\tau}$ denote the event that β_N does not hit $\partial\Lambda_{N-1}$. In that case $W^{\tau}_{N,\tau}$ does not hit $\partial\Lambda_N$. Clearly,

$$\mathbb{P}_N(\mathcal{V}_{N,\tau} \leq b) \geq \mathbb{P}_N(\mathcal{V}_{N,\tau} \leq b,\, C_N(\tau)).$$

We can now simply repeat the argument for the upper bound, using the same skeleton approach, the result being that

$$\lim_{\tau\to\infty} \frac{1}{\tau} \log \mathbb{P}_N\Big(\mathcal{V}_{N,\tau} \leq b \mid C_N(\tau)\Big) = -\bar{I}^{\downarrow}_{d,N}(b),$$

with $\bar{I}^{\downarrow}_{d,N}(b)$ given by the same variational formula, except that $\mathbb{R}^d$ is replaced by Λ_N with killing boundary.

We have

$$\lim_{\tau\to\infty} \frac{1}{\tau} \log \mathbb{P}_N(C_{N,\tau}) = -\lambda_N,$$

with λ_N the principal Dirichlet eigenvalue of Δ on Λ_N. It therefore follows that

$$\liminf_{\tau\to\infty} \frac{1}{\tau} \log \mathbb{P}_N(\mathcal{V}_{N,\tau} \leq b) \geq -\bar{I}^{\downarrow}_{d,N}(b) - \lambda_N.$$

But $\lim_{N\to\infty} \lambda_N = 0$. Moreover,

$$\lim_{N\to\infty} \bar{I}^{\downarrow}_{d,N}(b) = I^{\downarrow}_{d}(b)$$

by the same type of argument as sketched in the proof of Proposition 2. ■

OUFF, that was a long and intricate proof!

LECTURE 4

Sketch of the proof of the LDP for the capacity:
skeleton approach

§ SKELETON APPROACH FOR CAPACITY

The skeleton approach can also be used for the capacity. However, there are several important changes.

- For the volume the scale of the Brownian motion is $t^{1/d}$, while for the capacity it is $t^{1/(d-2)}$.

- The variational formula for the capacity has a more involved constraint than for the volume.

In what follows we sketch the main differences.

(1) COMPACTIFICATION

Note that $\mathsf{Cap}(A) \geq \mathsf{Cap}(TA)$ for any Borel set $A \subset \mathbb{R}^d$ and any map T that is a reflection with respect to a $(d-1)$-dimensional hyperplane.

By successive reflections of β with respect to the faces of $t^{1/(d-2)}\Lambda_N$, we get

$$\mathsf{Cap}(W_t) \geq \mathsf{Cap}(W_{N,t}),$$

where $W_{N,t}$ is the Wiener sausage at time t of a reflected Brownian motion on $t^{1/(d-2)}\Lambda_N$.

By Brownian scaling,

$$t^{-1}\mathsf{Cap}(W_{N,t}) =^{\mathsf{law}} \mathsf{Cap}(W^{\tau}_{N,\tau})$$

with

$$W^{\tau}_{N,\tau} = \cup_{s \in [0,\tau]} B_{r_\tau}(\beta_{N,s}),$$

where

$$\tau = t^{(d-4)/(d-2)}, \qquad r_\tau = \tau^{-1/(d-4)},$$

and β_N is a reflected Brownian motion on Λ_N. Putting $\mathcal{C}_{N,\tau} = \mathrm{Cap}(W^{\tau}_{N,\tau})$, we have

$$\mathbb{P}(\mathcal{C}_t \leq bt) \leq \mathbb{P}_N(\mathcal{C}_{N,\tau} \leq b),$$

where $\mathbb{P}_N$ is the law of β_N starting from 0.

Henceforth we suppress N from the notation.

(2) EXCURSION DECOMPOSITION

The same as before. The key players are

$$\mathbb{X}_{\tau,\epsilon} = \{\beta_{i\epsilon}\}_{i=1}^{\tau/\epsilon},$$

$$L_{\tau,\epsilon} = \frac{1}{\tau/\epsilon} \sum_{i=1}^{\tau/\epsilon} \delta_{(\beta_{(i-1)\epsilon},\beta_{i\epsilon})}.$$

Below we write $\mathbb{E}_{\tau,\epsilon}(\mathcal{C}_\tau)$ in terms of $L_{\tau,\epsilon}$.

(3) CONCENTRATION

LEMMA 1

For every $\delta > 0$,

$$\lim_{\epsilon\downarrow 0} \limsup_{\tau\to\infty} \frac{1}{\tau} \log \mathbb{P}\Big(|\mathcal{C}_\tau - \mathbb{E}_{\tau,\epsilon}(\mathcal{C}_\tau)| \geq \delta\Big) = -\infty.$$

Again use that $\mathcal{C}_\tau = \mathrm{Cap}(\cup_{i=1}^{\tau/\epsilon} W_i)$ is Lipschitz in the W_i's.

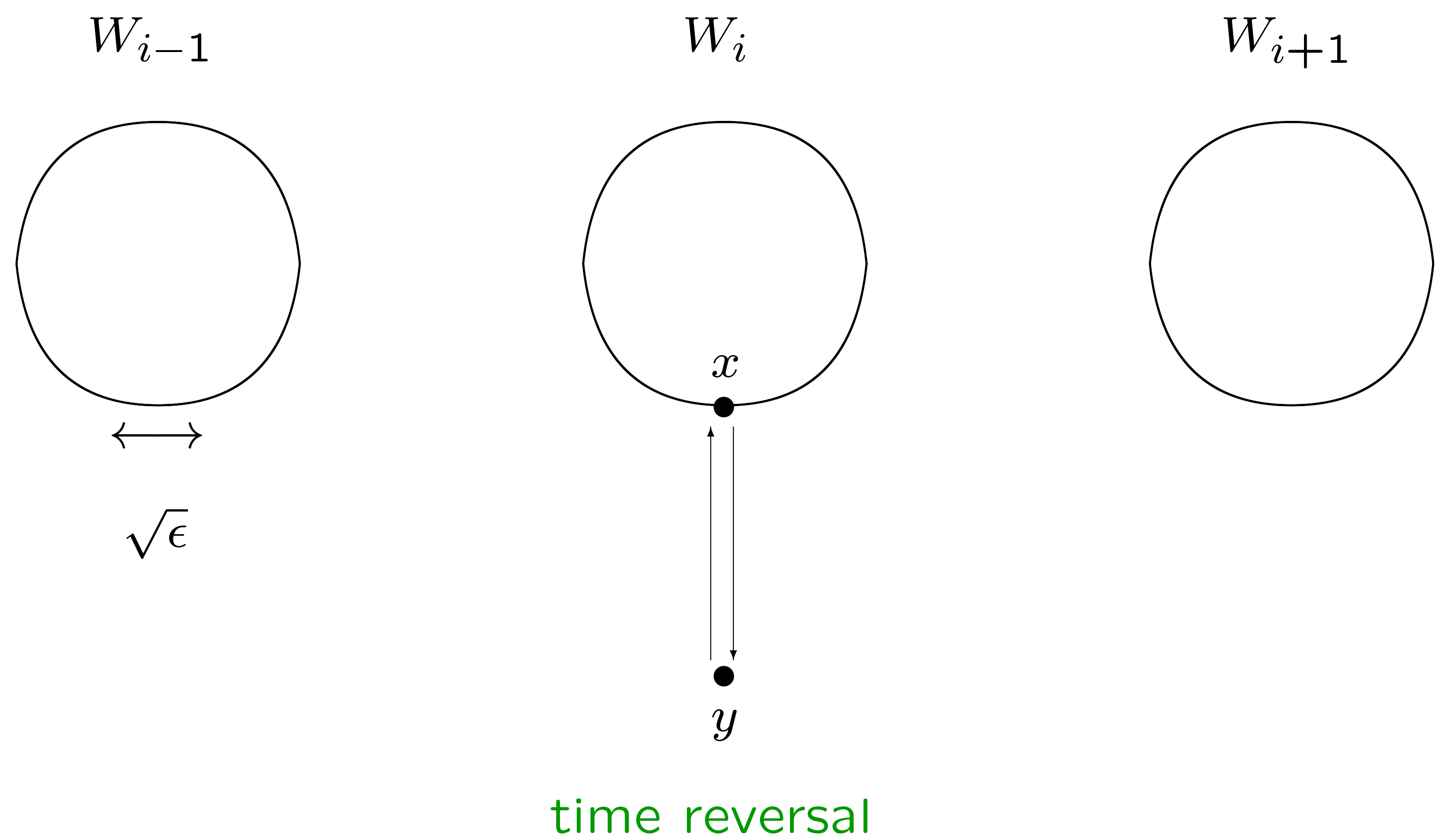
(4) DONSKER-VARADHAN LDP

Note that $W_\tau = \cup_{i=1}^{\tau/\epsilon} W_i$. The radius of W_i is $0 < r_\tau \ll 1$, which tends to zero as $\tau \to \infty$. Hence the union is close to being disjoint. Write

$$\mathcal{C}_\tau = \lim_{|y|\to\infty} \kappa_d |y|^{d-2}\, \bar{\mathbb{P}}_y(\bar{\tau}_{W_\tau} < \infty), \qquad (*)$$

where bar refers to an auxiliary Brownian motion. Write

$$\begin{aligned}
&\bar{\mathbb{P}}_y(\bar{\tau}_{W_\tau} < \infty) \\
&\sim^{!} \sum_{i=1}^{\tau/\epsilon} \int_{\partial W_i} \bar{\mathbb{P}}_y\Big(\bar{\tau}_{W_i} < \infty,\, \bar{\tau}_{W_\tau \setminus W_i} > \bar{\tau}_{W_i},\, \bar{\beta}_{\bar{\tau}_{W_i}} \in \mathrm{d}x\Big) \\
&= \sum_{i=1}^{\tau/\epsilon} \bar{\mathbb{P}}_y\Big(\bar{\tau}_{W_i} < \infty\Big) \\
&\qquad\qquad \int_{\partial W_i} \bar{\mathbb{P}}_y\Big(\bar{\tau}_{W_\tau \setminus W_i} > \bar{\tau}_{W_i},\, \bar{\beta}_{\bar{\tau}_{W_i}} \in \mathrm{d}x \;\Big|\; \bar{\tau}_{W_i} < \infty\Big).
\end{aligned}$$


W_{i-1}
W_i
W_{i+1}
x
y
$\sqrt{\epsilon}$
time reversal

But

$$\mathrm{Cap}(W_i) = \lim_{|y|\to\infty} \kappa_d |y|^{d-2}\, \bar{\mathbb{P}}_y(\bar\tau_{W_i} < \infty),$$

while, by time reversal,

$$\begin{aligned}
&\lim_{|y|\to\infty} \int_{\partial W_i} \bar{\mathbb{P}}_y\Big(\bar\tau_{W_\tau\setminus W_i} > \bar\tau_{W_i},\ \bar\beta_{\bar\tau_{W_i}} \in \mathrm{d}x \;\Big|\; \bar\tau_{W_i} < \infty\Big)\\
&=^{!} \int_{\partial W_i} p_{i,\partial W_\tau}(\mathrm{d}x)\ \bar{\mathbb{P}}_x\Big(\bar\tau_{W_\tau\setminus W_i} = \infty \;\Big|\; \bar\tau_{W_i} = \infty\Big)\\
&=^{\mathrm{def}}\ u_i(W_\tau),
\end{aligned}$$

where

$$p_{i,\partial W_\tau}(\mathrm{d}x) = \frac{h_{\partial W_\tau}(\mathrm{d}x)}{h_{\partial W_\tau}(\partial W_i)}, \qquad x \in \partial W_i,$$

with $h_{\partial W_\tau}$ the harmonic measure on ∂W_τ.

Combining this with $(*)$, we obtain

$$\mathcal{C}_\tau \sim^{!} \sum_{i=1}^{\tau/\epsilon} \mathsf{Cap}(W_i)\, u_i(W_\tau).$$

By Brownian scaling, we have

$$\begin{aligned}
\mathsf{Cap}(W_i) &= \mathsf{Cap}\left(\cup_{s\in[(i-1)\epsilon, i\epsilon]} B_{r_\tau}(\beta_s)\right) \\
&=^{\mathsf{law}} \mathsf{Cap}\left(\cup_{s\in[0,\epsilon]} B_{r_\tau}(\beta_s)\right) \\
&=^{\mathsf{law}} r_\tau^{d-2}\mathsf{Cap}\left(\cup_{s\in[0,\epsilon\, r_\tau^{-2}]} B_1(\beta_s)\right) \\
&\sim^{!} r_\tau^{d-2}\, c_d\, \epsilon\, r_\tau^{-2} = \frac{c_d}{\tau/\epsilon}, \quad \tau\to\infty,\ \mathbb{P}-a.s. \quad \forall\ i,
\end{aligned}$$

where in the last line we use the LLN.

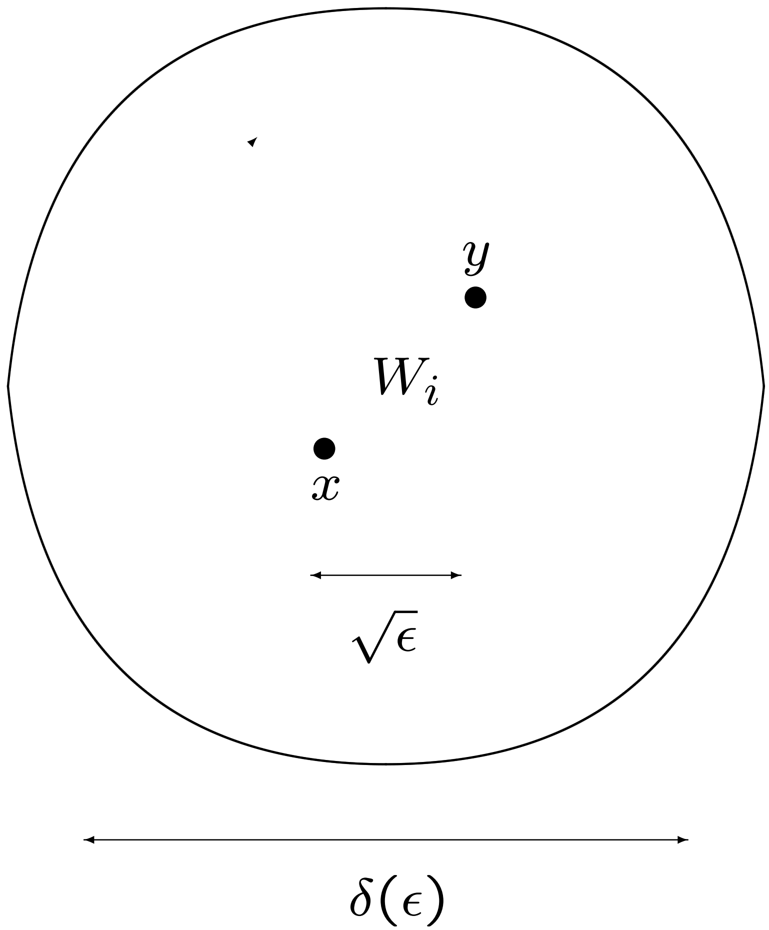
It follows that

$$\mathbb{E}_{\tau,\epsilon}(\mathcal{C}_\tau) \sim c_d \frac{1}{\tau/\epsilon} \sum_{i=1}^{\tau/\epsilon} \mathbb{E}_{\tau,\epsilon}(u_i(W_\tau))$$
$$\sim^{!} c_d \int_{\Lambda_N \times \Lambda_N} L_{\tau,\epsilon}(\mathrm{d}x, \mathrm{d}y)$$
$$\times \mathbb{E}_{\tau,\epsilon}\Big(\bar{\mathbb{P}}_{\frac{x+y}{2}}\Big(\bar{\tau}_{W_\tau \setminus B_{\delta(\epsilon)}(\frac{x+y}{2})} = \infty\Big)\Big),$$
$$\epsilon \downarrow 0, \ \mathbb{P} - a.s.$$

In the last line we use that if $(\beta_{(i-1)s}, \beta_{is}) = (x,y)$, then $|\bar{\beta}_{\bar{\tau}_{W_i}} - \frac{x+y}{2}| = O(\sqrt{\epsilon}\,)$ as $\epsilon \downarrow 0$, $\mathbb{P}_{\tau,\epsilon}$-a.s., while the condition $\bar{\tau}_{W_i} = \infty$ pushes $\bar{\beta}$ a distance

$$\sqrt{\epsilon} \ll \delta(\epsilon) \ll 1$$

away from W_i. Subsequently, $\bar{\beta}$ must avoid W_τ. But from distance $\delta(\epsilon)$ it is unlikely to hit $W_\tau \cap B_{\delta(\epsilon)}(x)$.



W_i is unlikely to exit the ball of radius $\delta(\epsilon)$ around $\frac{x+y}{2}$, while the constraint $\bar{\tau}_{W_i} = \infty$ pushes $\bar{\beta}$ out of this ball.

Our next observation is that, conditional on

$$L_{\tau,\epsilon} \sim \phi^2 \lambda \otimes \pi_\epsilon, \quad \tau \to \infty,\ \epsilon \downarrow 0,\ \lambda = \text{Lebesgue},$$

we have

$$\mathbb{E}_{\tau,\epsilon}\Big(\bar{\mathbb{P}}_{\frac{x+y}{2}}\Big(\bar{\tau}_{W_\tau \setminus B_{\delta(\epsilon)}(\frac{x+y}{2})} = \infty\Big)\Big)$$
$$\stackrel{!}{\sim} \bar{\mathbb{E}}_x\left(\exp\left[-\frac{c_d}{2d}\int_0^\infty \mathrm{d}s\, \phi(\bar{\beta}_s)^2\right]\right), \qquad \tau \to \infty,\ \epsilon \downarrow 0.$$

Indeed, β spends time $\tau\phi(x)^2\mathrm{d}x$ inside the volume element $\mathrm{d}x$. Hence $\mathrm{Cap}(W_\tau \cap \mathrm{d}x) = c_d\phi(x)^2\mathrm{d}x$. When $\bar{\beta}$ is inside $\mathrm{d}x$, it has a probability $\frac{1}{2d}\mathrm{Cap}(W_\tau \cap \mathrm{d}x)\, ds$ to hit W_τ during the time interval ds.

The right-hand side equals $u_\phi(x)$, which is the solution of the Schrödinger equation with potential $-\frac{c_d}{2d}\phi^2$ and with boundary condition $u(\infty) \equiv 1$.

(5) SHORT EXCURSION LIMIT

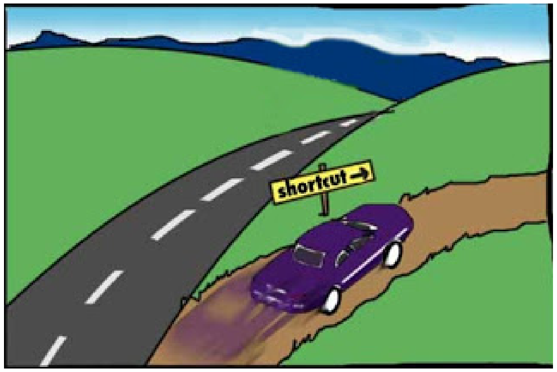
As we saw before,

$$\mathbb{P}\Big(L_{\tau,\epsilon} \sim \phi^2\lambda \otimes \pi_\epsilon\Big) \sim \exp\Big[-\tau I(\phi^2\lambda)\Big], \qquad \tau \to \infty,\ \epsilon \downarrow 0,$$

with

$$I(\phi^2\lambda) = \int_{\Lambda_N} |\nabla\phi|^2(x)\,\mathrm{d}x.$$

Combining the above observations, we get the analogue of Proposition 1: the LDP upper bound on Λ_N.



(6) DECOMPACTIFICATION

The variational formula for the rate function on Λ_N with reflecting boundary converges to the one on $\mathbb{R}^d$ as $N \to \infty$. Hence we also get the analogue of Proposition 2.

For the lower bound we can again use the killing boundary.

This completes the sketch of the proof of the downward LDP for the capacity. Many technical details were left out and are still under construction.


AAAHHHH!